Fuzzy Bounded and Continuous Linear Operators on Standard Fuzzy Normed Spacesa

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ABSTRACT

In this paper we introduce the definition of standard fuzzy normed space then we discuss several properties after we give an example to illustrate this notion. Then we define F-bounded operator as an introduction to define a standard fuzzy norm of an operator and if T is a linear operator from standard fuzzy normed space X into a standard fuzzy normed space Y we prove that T is continuous if and only if T is F-bounded.

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Key Words: standard fuzzy normed space, F-bounded linear operator, a fuzzy norm of an operator

INTRODUCTION

The theory of fuzzy sets was introduced by Zadeh in 1965[1]. Many authors have introduced the concept of fuzzy norm in different ways [2,3,4,5,6,7,11,12]. Cheng and Mordeson in 1994[8] defined fuzzy norm on a linear space whose associated fuzzy metric is of Kramosil and Mickalek type[9] as follows:

The order pair (X,N) is said to be a fuzzy normed space if X is a linear space and N is a fuzzy set on X ×]0,∞[ satisfying the following conditions for every x,y ∈ X and s,t ∈]0,∞[
(i) N(x,0) = 0, for all x ∈ X.
(ii)For all t>0, N(x,t) = 1 if and only if x = 0

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(iii) $N(\alpha x,t) = N(x,\frac{1}{\alpha t})$, for all $\alpha \neq 0$ and For all $t>0$.
(iv) For all $s, t>0$, $N(x+y,t+s) \geq N(x,t) \land N(y,s)$ where $\alpha \land \beta = \min\{\alpha, \beta\}$
(v) $\lim_{t \to 0} N(x,t) = 1$.

George and Veeramani in [10] introduced the definition of continuous $t$-norm. Bag and Samanta in [2] modified the definition of Cheng and Mordeson of fuzzy norm as follows:

The triple $(X,N,\ast)$ is said to be a fuzzy normed space if $X$ is a linear space, $\ast$ is a continuous $t$-norm and $N$ is a fuzzy set on $[0,\infty)$ satisfying the following conditions for every $x,y \in X$ and $s,t \in [0,\infty)$
(i) $N(x,0) = 0$, for all $x \in X$.
(ii) For all $t>0$, $N(x,t) = 1$ if and only if $x = 0$
(iii) $N(\alpha x,t) = N(x,\frac{1}{\alpha t})$, for all $\alpha \neq 0$
(iv) For all $s, t>0$, $N(x,t) \ast N(y,s) \leq N(x+y,t+s)$
(v) For $x \neq 0$, $N(x, \cdot):(0,\infty) \to [0,1]$ is continuous.
(vi) $\lim_{t \to 0} N(x,t) = 1$.

In this paper we introduce the definition of standard fuzzy normed space as a modification of the notion of fuzzy normed space due to Bag and Samanta. In section one we recall the definition of $t$-norm then we introduce the definition of standard fuzzy normed space after that we give an example then we prove that every ordinary norm induced a standard fuzzy norm define open ball, convergent sequence, open set, Cauchy sequence, F-bounded set and a continuous operator between two standard fuzzy normed spaces. Also we prove several properties for F-bounded operator.

**Standard fuzzy normed space**

**Definition 1.1:**[1]
Let $X$ be a nonempty set of elements, a fuzzy set $A$ in $X$ is characterized by a membership function, $\mu_A(x): X \to [0,1]$. Then we can write $A = \{(x, \mu_A(x)): x \in X, 0 \leq \mu_A(x) \leq 1\}$. Then $A$ is a continuous fuzzy set.

We now give an example of continuous fuzzy set

**Example 1.2:**[4]

Let $X = \mathbb{R}$ and let $A$ be a fuzzy set in $\mathbb{R}$ with membership function defined by :
$\mu_A(x) = \frac{1}{1+10x^2}$.

**Definition 1.3:**[10]
A binary operation $\ast$: $[0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm (or simply $t$-norm ) if for all $a, b, c, e \in [0,1]$ the following conditions hold:
1- $a \ast b = b \ast a$
2- $\ast 1 = a$
3- $(a \ast b) \ast c = a \ast (b \ast c)$
4- If $a \leq c$ and $b \leq e$ then $a \ast b \leq c \ast e$

**Example 1.4:**[10]
Define $a \ast b = a.b$, for all $a, b \in [0,1]$, where $a.b$ is the usual multiplication in $[0,1]$ then $\ast$ is a continuous $t$-norm.

**Example 1.5:**[10]
Define $a \ast b = \min\{a,b\}$ for all $a, b \in [0,1]$, it follows that $\ast$ is a continuous $t$-norm.

**Remark 1.6:**[10]
For any $a > b$, we can find $c$ such that $a * c \geq b$ and for any $d$ we can find $q$ such that $q * q \geq d$, where $a$, $b$, $c$, $d$ and $q$ belong to $(0,1)$. Now we introduce the basic definition in this paper

**Definition 1.7:**
Let $X$ be a linear space over field $\mathbb{K}$ and $*$ is a continuous t-norm and $N$ is a fuzzy set on $X$ satisfying:

(FN$_1$) $N(x) > 0$ for all $x \in X$.
(FN$_2$) $N(x) = 1$ if and only if $x = 0$.
(FN$_3$) $N(ax) = \frac{1}{|a|} N(x)$ for all $x \in X$ and $a \neq 0 \in \mathbb{K}$.
(FN$_4$) $N(x+y) \geq N(x) * N(y)$ for all $x, y \in X$.
(FN$_5$) $N(x)$ is a continuous fuzzy set.

Then the triple $(X, N, *)$ is called standard fuzzy normed space.

**Definition 1.8:**
Let $(X, N, *)$ be a standard fuzzy normed space. $N$ is called continuous fuzzy set if whenever $x_n \to x$ in $X$ then $N(x_n) \to N(x)$, that is $\lim_{n \to \infty} N(x_n) = N(x)$.

**Example 1.9:**
Let $X = \mathbb{Z}$, the set of integers, $a * b = a.b$ for all $a, b \in [0,1]$.

Define $N(x) = \begin{cases} \frac{1}{|x|} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Then $(X, N, *)$ is standard fuzzy normed space.

**Proposition 1.10:**
Let $(X, \| . \|)$ be an ordinary normed space with $\| x \|$ is an integer for all $x \in X$. Define $N_{\| . \|}(x) = \begin{cases} \frac{1}{|x|} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ and $a * b = a.b$ for all $a, b \in [0,1]$. Then $(X, N_{\| . \|}, *)$ is standard fuzzy normed space induced by $\| . \|$.

**Proof:**
(FN$_1$) Since $\| x \| > 0$ for all $x \in X$ then $N_{\| . \|}(x) > 0$ for all $x \in X$.
(FN$_2$) $N_{\| . \|}(x) = 1$ if and only if $x = 0$.
(FN$_3$) Let $a \neq 0 \in K$ then for all $x \in X$ we have $N_{\| . \|}(ax) = \frac{1}{|ax|} = \frac{1}{|a|} \frac{1}{|x|} = \frac{1}{|a|} N_{\| . \|}(x)$.
(FN$_4$) $N_{\| . \|}(x+y) = \frac{1}{|x+y|} \geq \frac{1}{|x|} \frac{1}{|y|} = N_{\| . \|}(x) * N_{\| . \|}(y)$.
(FN$_5$) Let $(x_n)$ be a sequence in $X$ such that $x_n \to x$ that is $\lim_{n \to \infty} x_n = x$.

Now, $\lim_{n \to \infty} N_{\| . \|}(x_n) = \lim_{n \to \infty} \frac{1}{|x_n|} = \frac{1}{|x|} = N_{\| . \|}(x)$.

Therefore $N_{\| . \|}$ is continuous fuzzy set. Hence $(X, N_{\| . \|}, *)$ is standard fuzzy normed space.

**Definition 1.11:**
Let $(X, N, *)$ be a standard fuzzy normed space, we define $B(x, r) = \{ y \in X : N(x) > (1-r) \}$ then $B(x, r)$ is called an open ball with center $x \in X$ and radius $0 < r < 1$. 


Definition 1.12:
A sequence \((x_n)\) in a standard fuzzy normed space \((X, N, \ast)\) is said to be converge to a point \(x \in X\) if \(0 < \varepsilon < 1\) is given, there exists a positive number \(K\) such that, \(N(x_n - x) > (1 - \varepsilon)\) for all \(n \geq K\).

Theorem 1.13:
A sequence \((x_n)\) in a standard fuzzy normed space \((X, N, \ast)\) is converge to a point \(x \in X\) if and only if \(\lim_{n \to \infty} N(x_n - x) = 1\).

Proof:
Suppose that the sequence \((x_n)\) converges to \(x\) then for given any \(0 < r < 1\) there is a positive number \(K\) such that \(N(x_n - x) > (1 - r)\) for all \(n \geq K\) and hence \(1 - N(x_n - x) < r\). Therefore \(N(x_n - x)\) converges to \(1\) as \(n\) tends to \(\infty\). The proof of the converse is similar hence is omitted.

Lemma 1.14:
Let \((X, N, \ast)\) be a standard fuzzy normed space. Then \(N(x - y) = N(y - x)\) for all \(x, y \in X\).

Proof:
\[ N(x - y) = N(-1)(y - x) = \frac{1}{|-1|} N(y - x) = N(y - x). \]

Definition 1.15:
A subset \(A\) of a standard fuzzy normed space \((X, N, \ast)\) is said to be open if it contains a ball about each of its points. A subset \(B\) of \(X\) is said to be closed if its complement is open that is \(B^c = X - B\) is open.

The proof of the following theorem is easy, hence it is omitted.

Theorem 1.16:
Every open ball in a standard fuzzy normed space \((X, N, \ast)\) is an open set.

Definition 1.17:
Let \((X, N, \ast)\) be a standard fuzzy normed space and let \(A \subseteq X\) then the closure of \(A\) is denoted by \(\overline{A}\) or \(cl(A)\) and is defined to be the smallest closed set contains \(A\).

Lemma 1.18:
Let \(A\) be a subset of a standard fuzzy normed space \((X, N, \ast)\). Then \(a \in \overline{A}\) if and only if there is a sequence \((a_n)\) in \(A\) such that \(a_n \to a\).

Proof:
Let \(a \in \overline{A}\), if \(a \in A\) then we take sequence of that type \((a, a, a, \ldots, a, \ldots)\). If \(a \notin A\), then it is a limit point of \(A\). Hence we construct the sequence \((a_n)\) \(\subseteq A\) by \(N(a_n - a) > 1 - \frac{1}{n}\) for each \(n = 1, 2, 3, \ldots\)

The ball \(B(a, \frac{1}{n})\) contains \(a_n \in A\) and \(a_n \to a\) because \(\lim_{n \to \infty} N(a_n - a) = 1\).

Conversely if \((a_n)\) in \(A\) and \(a_n \to a\) then \(a \in A\), or every neighborhood of \(a\) contains points \(a_n \neq a\), so that \(a\) is a point of accumulation of \(A\), hence \(a \in \overline{A}\) by using the definition of the closure.

Definition 1.19:
A sequence \((x_n)\) in a standard fuzzy normed space \((X, N, \ast)\) is said to be Cauchy if for each \(0 < \varepsilon < 1\) there is a positive number \(K\) such that \(N(x_n - x_m) > (1 - \varepsilon)\) for all \(n, m \geq K\).

The proof of the following theorem is easy, hence it is omitted.

Theorem 1.20:
In a standard fuzzy normed space every convergent sequence is Cauchy.

Definition 1.21:
Let \((X,N,\ast)\) be a standard fuzzy normed space. A subset \(A\) of \(X\) is said to be F-bounded if there exists a real number \(r, 0 < r < 1\) such that, \(N(x) > (1 - r)\), for all \(x \in A\).

**Definition 1.22:**
Let \((X,N_X,\ast)\) and \((Y,N_Y,\ast)\) be two standard fuzzy normed spaces and \(A \subseteq X\). The operator \(f:A \rightarrow Y\) is said to be continuous at \(a \in A\), if for every \(0 < \varepsilon < 1\), there exists some \(0 < \delta < 1\), such that \(N_Y(f(x) - f(a)) > (1 - \varepsilon)\) whenever \(x \in A\) satisfying \(N_X(x-a) > (1 - \delta)\). If \(f\) is continuous at every point of \(A\), then it is said to be continuous on \(A\).

**Fuzzy Bounded and Continuous Linear Operator**

**Definition 2.1:**
Let \((X,N_X,\ast)\) and \((Y,N_Y,\ast)\) be two fuzzy normed spaces and \(T: D(T) \rightarrow Y\) be a linear operator, where \(D(T) \subseteq X\). The operator \(T\) is said to be F-bounded if there is a real number \(c, 0 < c < 1\) such that for all \(x \in D(T), N_Y(Tx) \geq (1-c) N_X(x) \). ...

**Remark 2.2**
1- Formula (2.1) shows that F-bounded linear operator maps F-bounded sets in \(D(T)\) onto F-bounded sets in \(Y\).
2- What is the largest possible \((1-c)\) such that equation (2.1) still holds for all \(x \in D(T)?\). By division we have \(\frac{N_Y(Tx)}{N_X(x)} \geq (1-c)\) and this shows that \((1-c)\) must be at least as big as the infimum of the \(N_Y(Tx)\) over \(D(T)\) - \(\{0\}\). Hence the answer to our question is that the smallest possible \((1-c)\) in (2.1) is that infimum. This quantity is denoted by \(N(T)\). Thus
\[ N(T) = \inf_{x \in D(T)} \frac{N_Y(Tx)}{N_X(x)} \] ...

**Lemma 2.3**
Let \(T: D(T) \rightarrow Y\) be fuzzy bounded linear operator from a standard fuzzy normed space \((X,N_X,\ast)\) into a standard fuzzy normed space \((Y,N_Y,\ast)\) then
(i) An alternative formula for the norm of \(T\) is
\[ N(T) = \inf_{x \in D(T)} N_Y(Tx) \] ...
(ii) The norm defined by (2.2) is a standard fuzzy normed space

**Proof:**
(i) We put \(a = N(x)\) and set \(y = ax\). Then
\[ N_X(y) = N_X(ax) = \frac{1}{|a|} N_X(x) = \frac{1}{N_X(x)} N_X(x) = 1 \] and since \(T\) is linear equation (2.2) gives
\[ \inf_{x \in D(T)} \frac{N_Y(Tx)}{N_X(x)} = \inf_{x \in D(T)} \frac{N_Y(Tx)}{ax} = \inf_{x \in D(T)} N_Y(T(ax)) \]
Writing \(x\) for \(y\) on the right, we have (2.4).
(ii) (FN1) \(N_Y(Tx) > 0\) and \(N_X(x) > 0\) implies \(\frac{N_Y(Tx)}{N_X(x)} > 0\). Hence \(N(T) > 0\). 
(FN2) \(N(T) = 1 \iff \inf_{x \in D(T)} N_Y(Tx) = 1 \iff N_Y(Tx) = 1 \iff Tx = 0 \iff T = 0\).
(FN_3) \( N(\alpha T) = \inf_{x \in D(T)} N_Y(\alpha Tx) = \frac{1}{|\alpha|} \inf_{x \in D(T)} N_Y(Tx) = \frac{1}{|\alpha|} N(T) \).

(\text{FN}_4) \( N(T_1 + T_2) = \inf_{x \in D(T_1) \cap D(T_2)} N_Y[(T_1 + T_2)(x)] \)
\[
\geq \inf_{x \in D(T_1)} N_Y(T_1(x) + T_2(x))
\geq \inf_{x \in D(T_1)} N_Y(T_1(x)) \cdot \inf_{x \in D(T_2)} N_Y(T_2(x))
\geq N(T_1) \cdot N(T_2)
\]

(\text{FN}_5) Since \( N_Y \) is continuous so \( N(T) \) is continuous.

Before we consider general properties of F-bounded linear operators, let us take a look at some typical examples, so that we get a better feeling for the concept of a F-bounded linear operator.

**Example 2.4:** Let \( X \) be the vector space of all polynomials on \( J = [0,1] \) with norm given by \( \|x\| = \max |x(t)|, t \in J \) where \( |x(t)| \) is an integer then \( (X,N_{\|\|,*}) \) is a standard fuzzy normed space where \( N_{\|\|}(x) = \frac{1}{\|x\|} \) if \( x \neq 0 \) and \( N_{\|\|}(0) = 1 \) also \( a \ast b = a \cdot b \) \( \forall a, b \in J = [0,1] \).

Let \( T: X \to X \) defined by:
\[
T(x(t)) = x'(t).
\]

\( T \) is linear but not F-bounded. Indeed \( x_n(t) = t^n, n=1, 2, \ldots \) so \( \|x_n\| = 1 \), then \( N_{\|\|}(x_n) = 1 \) where \( n \in \mathbb{N} \). Now, \( Tx_n(t) = nt^{n-1} \) so \( \|Tx_n\| = n \) which implies that \( N_{\|\|}(Tx_n) = \frac{1}{n} \) so
\[
N(Tx_n) = \frac{1}{n} \leq N(x_n).
\]

Since \( n \in \mathbb{N} \) is arbitrary, this shows that there is no \( r, 0 < r < 1 \) such that
\[
N(Tx_n) \geq (1-r)N(x_n).
\]
From this we conclude that \( T \) is not F-bounded.

**Example 2.5:** Consider \( C[0,1] \) with \( \|x\| = \max |x(t)|, t \in J = [0,1] \) with \( |x(t)| \) is an integer. Then \( (C[0,1],N_{\|\|,*}) \) is a standard fuzzy normed space where
\[
N_{\|\|}(x) = \frac{1}{\|x\|}, N_{\|\|}(0) = 1, \text{ and } a \ast b = a \cdot b \text{ for all } a, b \in [0,1].
\]
Define \( T: C[0,1] \to C[0,1] \) by
\[
T(x) = \int_0^1 k(t,s)x(s)ds
\]
where \( k(t,s) \) is continuous on \( J \times J \) and \( k(t,s) \) is bounded say \( |k(t,s)| \leq k_0 \) for all \( (t,s) \in J \times J \) where \( k_0 \in \mathbb{R} \). This operator is linear and F-bounded. Now
\[
\|Tx(t)\| \leq \max |x(t)| = \|x(t)\|, t \in J.
\]

Hence, \( \|y\| = \|Tx\| = \max \int_0^1 k(t,s)x(s)ds \leq \max \int_0^1 |k(t,s)||x(s)|ds \leq k_0\|x\|. \)

Therefore
\[
N_{\|\|}(Tx) = \frac{1}{\|Tx\|} \geq \frac{1}{k_0} \cdot \frac{1}{\|x\|} = \frac{1}{k_0} N_{\|\|}(x).
\]

Put \( \frac{1}{k_0} = (1-r) \) for some \( r, 0 < r < 1 \), we get \( N_{\|\|}(Tx) \geq (1-r) N_{\|\|}(x) \).
Hence \( T \) is F-bounded.

Operators are mappings, so that the definition of continuity applies to them as follows: Let \( T: D(T) \to Y \) be any operator, not necessarily linear, where \( D(T) \subset X \) and \( X \) and \( Y \) are standard fuzzy normed spaces. The operator \( T \) is called continuous at \( x_0 \in D(T) \) if for every \( 0 < \varepsilon < 1 \) there is a \( 0 < \delta < 1 \) such that \( N_Y(Tx - Tx_0) > (1 - r) \) for all \( x \in D(T) \) satisfying \( N_X(x - x_0) > (1 - \delta) \). \( T \) is continuous if \( T \) is continuous at every \( x \in D(T) \).

**Theorem 2.6:** Let \( T: D(T) \to Y \) be a linear operator, where \( D(T) \subset X \) and \( (X,N_{\|\|,*}) \)
\( (Y,N_Y,*)) \) are standard fuzzy normed spaces. Then:
(i) T is continuous if and only if T is F-bounded.
(ii) If T is continuous at a single point, it is continuous.

Proof of (i):
We assume T is F-bounded and consider any $x_o \in \text{D}(T)$. Let any $0 < \varepsilon < 1$ be given. Then, since T is linear, for every $x \in \text{D}(T)$ such that $N_x(x - x_o) > (1 - \delta)$, we obtain

$$N_y(Tx - Tx_o) = N_y[T(x - x_o)] \geq N(T) \cdot N_x(x - x_o) \geq N(T) \cdot (1 - \delta) = (1 - \varepsilon).$$

Since $x_o \in \text{D}(T)$ was arbitrary, this shows that T is continuous.

Conversely, assume that T is continuous at an arbitrary $x_o \in \text{D}(T)$ Then, given any $0 < \varepsilon < 1$, there is a $0 < \delta < 1$ such that $N_y(Tx - Tx_o) \geq (1 - \varepsilon)$ for all $x \in \text{D}(T)$ satisfying $N_x(x - x_o) > (1 - \delta)$.

We now take $y \neq 0$ in $\text{D}(T)$ and set $x = x_o + \frac{N_x(y)}{(1 - \delta)} \cdot y$. Then $N_x(x - x_o) = \frac{N_x(y)}{(1 - \delta)} \cdot y$.

Hence $N_y(Tx - Tx_o) = \frac{(1 - \delta)}{N_x(y)} N_y(y) = (1 - \delta)$. Now

$$N_y(Tx - Tx_o) = N_y(T(x - x_o)) = N_y\left[T\left(\frac{N_x(y)}{(1 - \delta)} \cdot y\right)\right] = \frac{(1 - \delta)}{N_x(y)} N_y(Ty) \quad \text{and (2.5) implies}$$

$$\frac{(1 - \delta)}{N_x(y)} N_y(Ty) \geq (1 - \varepsilon) \quad \text{implies} \quad N_y(Ty) \geq \frac{(1 - \delta)}{(1 - \delta)} N_y(y).$$

This can be written $N_y(Ty) \geq (1 - c) N_y(y)$, where $(1 - c) = \frac{(1 - \delta)}{(1 - \delta)}$ and this shows that T is F-bounded.

Proof of (ii):
Continuity of T at a point implies F-boundedness of T by the second part of the proof of (i), which in turn implies continuity of T by (i).

Corollary 2.7
Let $(X, N_X, *)$ be standard fuzzy normed spaces and let $T: \text{D}(T) \to Y$ be a F-bounded linear operator. Then:

(i) If $x_n \to x$ in $\text{D}(T)$ then $(x_n) \in \text{K}(T)$.

(ii) The null space $K(T)$ is closed, where $K(T) = \{x \in \text{D}(T) : T(x) = 0\}$.

Proof of (i):
Since T is F-bounded, $N(T) \geq (1 - r)$ for some $0 < r < 1$, and since $x_n \to x$ given $0 < s < 1$ there is a positive number K such that $N_x(x_n - x) > (1 - s)$. Now, by Remark (1.6) there is $(1 - \varepsilon) \in (0, 1)$ such that $(1 - \varepsilon) * (1 - s) > (1 - \varepsilon)$. Now

$$N_y(Tx_n - Tx) = N_y(T(x_n - x)) \geq N(T) * N_x(x_n - x) > (1 - r) * (1 - s) > (1 - \varepsilon)$$

for all $n \geq k$. Hence $Tx_n \to Tx$.

Proof of (ii):
For every $x \in \overline{\text{K}(T)}$ there is a sequence $(x_n)$ in $\text{K}(T)$ such that $x_n \to x$ by Lemma (1.18). Hence $Tx_n \to Tx$ by part (i) of this corollary. Also $Tx = 0$ since $Tx_n = 0$. So that $x \in \text{K}(T)$. Since $x \in \overline{\text{K}(T)}$ was arbitrary $\text{K}(T)$ is closed.

Theorem 2.8
Let $(X, N_X, *)$ be a standard fuzzy normed space and let $(Y, N_Y, *)$ be a Banach space.
Let $T: \text{D}(T) \to Y$ be a F-bounded linear operator, where $\text{D}(T) \subset X$. Then T has an extension $\hat{T}: \overline{\text{D}(T)} \to Y$ where $\hat{T}$ is a F-bounded linear operator of norm $N(\hat{T}) = N(T)$.

Proof:
We consider any $x \in \overline{\text{D}(T)}$. By Lemma (1.18) there is a sequence $(x_n)$ in
D(T) such that \( x_n \to x \). Since T is linear and F-bounded, we have \( N(T) \geq (1-r) \) for some \( 0 < r < 1 \) and since \( x_n \to x \) for any given \( 0 < s < 1 \) there is a number \( K \) such that \( N(x_n - x) > (1-s) \) for all \( n \geq K \). Hence by Remark (1.6), there is \( (1-\varepsilon) \in (0,1) \) such that \( (1-r) * (1-s) > (1-\varepsilon) \), for all \( n,m \geq k \). This shows that \( \{Tx_n\} \) is Cauchy. By assumption Y is Banach so that \( \{Tx_n\} \) converges, say, \( Tx_n \to y \). We define \( \overline{T} : D(T) \to Y : \overline{T}x = y \). We show that this definition is independent of the particular choice of a sequence in \( D(T) \) converging to \( x \).

Clearly, \( \overline{T} \) is linear and \( \overline{T}x = Tx \) for every \( x \in D(T) \), so that \( \overline{T} \) is an extension of \( T \). We now use \( N_y(Tx_n) \geq N(T) * N_x(x_n) \) and let \( n \to \infty \).

Then \( Tx_n \to y = \overline{T}x \). Since \( x \to N_y(x) \) defines a continuous operator \( \overline{T} \), we thus obtain \( N_y(\overline{T}x) \geq N(T) * N_x(x). \) Hence \( \overline{T} \) is F-bounded and \( N(\overline{T}) \geq N(T) \). Of course, \( N(\overline{T}) \leq N(T) \) because the fuzzy norm being defined by an infimum, cannot decrease in an extension. Together we have \( N(\overline{T}) = N(T) \).

REFERENCES