On Arps – Closed Sets in Topological Spaces

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Received on: 4/10/2012 & Accepted on: 3/10/2013

ABSTRACT
In this paper, we introduce a new class of closed sets which is called $\alpha rps$ – closed sets in topological spaces and we given the relationships of these sets with some other sets. Also, we study some of their properties. Further, will be introduce and study new type of spaces namely $T_{\alpha rps}$ – space and new type of continuous functions which are ( $\alpha rps$-continuous functions , $\alpha rps$- irresolute functions and strongly $\alpha rps$-continuous functions ) , we introduce several properties of these functions are proved.

INTRODUCTION

emi – open sets , regular open sets , $\alpha$-open and preopen sets have been introduced and investigated be Levin [15],Stone[25], Njastad [22] and Mashhour [20] respectively. In 1970, Levin [16] introduced generalized closed sets and studied their basic properties. After the introduction of generalized closed sets there are many research papers which deal with different types of generalized closed sets. Arya[3] , Bhattachary and Iahiri [6], Maki et a [17, 18 ] introduced generalized semi – closed sets, semi- generalized closed sets, $\alpha$-generalized closed sets and generalized $\alpha$- closed sets respectively. Also Dontchev [10], Maki et a [19],Ganambal [13],Palaniappan and Rao[23]. Nagaven and Ganster[11],they also introduced and investigated generalized semi- preclosed sets ,generalized preclosed, gp- sets , generalized pre regular closed sets, weakly generalized closed, regular generalized- closed sets and generalized b- closed sets.


https://doi.org/10.30684/etj.32.2B.10
2412-0758/University of Technology-Iraq, Baghdad, Iraq
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In this paper a new type of closed sets called \( \alpha \)-closed sets are introduced and its properties are studied. Applying these sets, we obtain a new space which is namely \( T_{\alpha} \)-spaces. Further we study ( \( \alpha \)-continuous, \( \alpha \)-irresolute and strongly \( \alpha \)-continuous) functions and we proved some their properties. Throughout this paper \( (X, \tau) \), \( (Y, \sigma) \) and \( (Z, \mu) \) (or simply \( X, Y \) and \( Z \)) represent non-empty topological spaces. For a subsets \( A \) of a spaces \( X \), \( \text{cl}(A) \), \( \text{int}(A) \) and \( A^c \) denote the closure of \( A \), the interior of \( A \) and the complement of \( A \) respectively.

**PRELIMINARIES**

Some definition and basic concepts have been given in this section.

**Definition (2-1):** A subset \( A \) of a space \( X \) is said to be:

1. **semi-open** [15] if \( A \subseteq \text{cl}(\text{int}(A)) \) and semi-closed if \( \text{int}(\text{cl}(A)) \subseteq A \).
2. **\( \alpha \)-open set** [22] if \( A \subseteq \text{int}(\text{cl}(t(A))) \) and \( \alpha \)-closed set if \( \text{cl}(\text{int}(\text{cl}(A))) \subseteq A \).
3. **Preopen** [20] if \( A \subseteq \text{int} \left( \text{cl}(\text{int}(A)) \right) \) and preclosed if \( \text{cl}(\text{int}(A)) \subseteq A \).
4. **semi-preopen set** [1] if \( A \subseteq \text{int}(\text{cl}(A)) \) and semi-preclosed if \( \text{int}(\text{cl}(\text{int}(A))) \subseteq A \).
5. **b-open set** [2] if \( A \subseteq \text{int}(\text{cl}(A)) \cap \text{cl}(\text{int}(A)) \) and b-closed set if \( \text{int}(\text{cl}(A)) \cap \text{cl}(\text{int}(A)) \subseteq A \).
6. **regular open** [25] if \( A = \text{int}(\text{cl}(A)) \) and regular closed if \( A = \text{cl}(\text{int}(A)) \).
7. **regular \( \alpha \)-open** [27] if there is a regular open set \( U \) such that \( U \subseteq A \subseteq \alpha \text{cl}(U) \).

The semi-closure (resp. \( \alpha \)-closure, resp. pre-closure, resp. semi-pre closure, resp. b-closure) of a sub set \( A \) of \( X \) is the intersection of all semi-closed (resp \( \alpha \)-closed, resp. pre-closed, resp. semi-pre closed, resp. b-closed) sets containing \( A \) and denoted by \( \text{scl}(A) \) (resp \( \alpha \text{cl}(A) \), resp. \( \text{pc}(A) \), resp. \( \text{spcl}(A) \), resp. \( \text{bcl}(A) \)). Clearly, \( \text{bcl}(A) \leq \text{cl}(A) \) and \( \text{spcl}(A) \leq \text{pcl}(A) \leq \alpha \text{cl}(A) \leq \text{cl}(A) \).

**Remark (2-2):** In [2], [15], [22], it has been proved that:

(i) Every open set in a space \( X \) is an \( \alpha \)-open (resp. preopen, semi-preopen and b-open sets). Also every closed set is \( \alpha \)-closed (resp. preclosed, semi-preclosed and b-closed set).

(ii) Every \( \alpha \)-open set in a space \( X \) is a preopen (resp. semi-preopen, semi-open and b-open) set. Also every \( \alpha \)-closed set in space \( X \) is a preclosed (resp. semi-preclosed, semi-closed and b-closed) set.

(iii) The union of any family of \( \alpha \)-open sets is \( \alpha \)-open set and the intersection of any finite sets of \( \alpha \)-closed sets is \( \alpha \)-closed.

**Definition (2-3):** A sub set \( A \) of a space \( X \) is said to be:

1. **Generalized closed set** (briefly, \( g \)-closed) [16] if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is an open set in \( X \).
2. **Generalized semi-closed set** (briefly, \( gs \)-closed) [3] if \( \text{scl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is an open set in \( X \).
3. **semi-generalized closed set** (briefly, \( sg \)-closed) [6] if \( \text{scl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is a semi-open set in \( X \).
4. **generalized \( \alpha \)-closed set** (briefly, \( g\alpha \)-closed) [18] if \( \alpha \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is an \( \alpha \)-open set in \( X \).
5- \textbf{α-g} - generalized closed set (briefly, α g- closed) [17] if \( \text{acl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is an open set in \( X \).

6- \textbf{generalized pre closed set} (briefly, gp- closed) [19] if \( \text{pcl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is an open set in \( X \).

7- \textbf{generalized semi-pre closed set} (briefly, gsp- closed) [10] if \( \text{spcl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is a regular open set in \( X \).

8- \textbf{generalized pre regular closed set} (briefly, gpr- closed) [13] if \( \text{pcl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is a regular open set in \( X \).

9- \textbf{regular generalized closed set} (briefly, rg- closed) [23] if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is a regular open set in \( X \).

10- \textbf{regular weakly generalized closed set} (briefly, rwg- closed) [21] if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is a regular open set in \( X \).

11- \textbf{regular generalized α-closed set} (briefly, rgα-closed) [21] if \( \text{cl}(\text{int}(A)) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is a regular \( α \)-open set in \( X \).

12- \textbf{generalized b-closed set} (briefly, gb- closed) [13] if \( \text{bcl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is an open set in \( X \).

13- \textbf{Pre- semi closed} [29] if \( \text{spcl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is a regular open set in \( X \).

14- \textbf{regular pre-semi closed} (briefly, rps-closed) [24] if \( \text{spcl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is a regular open set in \( X \).

The complement of a g-closed (resp. gs-closed , sg-closed , gα-closed , αg-closed , gp-closed , gsp-closed, gpr-closed , rg-closed , rwg-closed , gb- closed) sets is called a g-open (resp. gs-open , sg-open , gα-open , αg-open , gp-open , gsp-open, gpr-open , rg-open , rwg-open , gb-open) sets , also the complement of rps- closed is called \textit{rps-open set}.

**Remark (2-4):** In [4],[12],[14] and [27] it has been proved that:
1-Every sg-closed set is a(gs-closed , gp-closed and gb-closed) set respectively.
2-Every gs-closed set is a gsp-closed set.
3-Every gb-closed set is a (sg-closed set, gs-closed set and gsp-closed set) Respectively.
4-Every ga-closed set is a (ag-closed , gp-closed , gpr-closed and rga-closed) set .
5-Every ag-closed set is a gp-closed set and gpr-closed set.
6-Every gp-closed set is a gpr-closed set .
7-Every rgα-closed set is an rg-closed set and rwg-closed set.

**Remark (2-5):** In [24] , it has been proved that :

(i) Every open set is rps-open. Also every closed set in \( X \) is rps-closed.

(ii) Every semi- open set is rps-open. Also every semi- closed set in \( X \) is rps-closed.

(iii) Every α- open set is an rps-open . Also every α- closed set in \( X \) is an rps-closed.

(iv) Every semi-pre open set is an rps-open . Also every semi-pre closed set in \( X \) is an rps-closed. But the converse of remark(2-5) need not be true in general .

**Example(2-1):** Let \( X = \{a,b,c,d\} \) , \( \tau = \{X,\emptyset,\{a\},\{b\},\{a,b\},\{b,c\},\{a,b,c\}\} \). Then

(i) \( \{b\,\{c\} \) is an rps-closed set but is not closed , α-closed and preclosed set in \( (X ,\tau) \).

Also, \( \{b,\{c\} \subseteq \{a,d\} \) is not open , α-open and preopen set in \( (X ,\tau) \).

(ii) \( \{a,b,d\} \) is an rps closed set in \( (X ,\tau) \) but not semi-closed and semi-preclosed ,and \( \{a,b,d\} \subseteq \{c\} \) is an rps open set but is not semi-open and semi-preopen.

**Definition (2-6):** A topological space \( X \) is said to be:
1- T\(^{1/2}\) – spaces [23] if every rg- closed sets is closed.
2- A T\(_b\) – space [8] if every gs- closed sets is closed.
3- An \(\alpha T_b\) -spaces [7] if every ag- closed sets is closed.
4- A T\(_{1/2}\) – space [16] if every g- closed sets is closed.

**Definition (2-7),[15]:** A function \(f : (X, \tau) \to (Y, \sigma)\) is said to be continuous function if the inverse image of each open( closed) set in Y is an open(closed) set in X.

**Definition (2-8):** A function \(f : (X, \tau) \to (Y, \sigma)\) from topological space X in to topological space Y is said to be:
1- generalized continuous (briefly, g- continuous) [5] if \(f^{-1}(A)\) is a g- closed set in X for every closed set A in Y.
2- generalized semi continuous (briefly, gs- continuous) [9] if \(f^{-1}(A)\) is a gs- closed set in X for every closed set A in Y.
3- semi- generalized continuous (briefly, sg- continuous) [26] if \(f^{-1}(A)\) is a sg- closed set in X for every closed set A in Y.
4- \(\alpha\)- generalized continuous (briefly, \(\alpha g\)- continuous) [17] if \(f^{-1}(A)\) is an \(\alpha g\)- closed set in X for every closed set A in Y.
5- Generalized \(\alpha\)- continuous (briefly, \(g\alpha\)- continuous) [7] if \(f^{-1}(A)\) is a \(g\alpha\)- closed set in X for every closed set A in Y.
6- Generalized pre- continuous (briefly, gp- continuous) [14] if \(f^{-1}(A)\) is a gp- closed set in X for every closed set A in Y.
7- Generalized semi pre- continuous (briefly, gsp- continuous) [16] if \(f^{-1}(A)\) is a gsp- closed set in X for every closed set A in Y.
8- Generalized preregular- continuous (briefly , gpr- continuous ) [12] if \(f^{-1}(A)\) is a gpr- closed set in X for every closed set A in Y.
9- Regular generalized- continuous (briefly, rg- continuous ) [23] if \(f^{-1}(A)\) is an rg- closed set in X for every closed set A in Y.
10- Regular weakly generalized- continuous (briefly , rwg- continuous ) [21] if \(f^{-1}(A)\) is an rwg- closed set in X for every closed set A in Y.
11- Regular generalizede-continuous (briefly, rga- continuous ) [28] if \(f^{-1}(A)\) is a rga- closed set in X for every closed set A in Y.
12- Generalized b- continuous (briefly , gb- continuous ) [4] if \(f^{-1}(A)\) is a gb- closed set in X for every closed set A in Y.

**\(\alpha RPS – CLOSED SETS IN TOPOLOGICAL SPACES :\)**

In this section, we introduce a new type of closed sets namely arps-closed sets in topological spaces and study some of their properties.

**Definition (3-1):** A subset A of a topological spaces X is said to be arps-closed set if acl(A) \(\subseteq\) U whenever A \(\subseteq\) U and U is an rps-open set in X. The complement of an arps- closed set is called arps- open and the class of all arps- closed (resp. arps-open) subset of X is denoted by arPSC(X) (resp. arPSO(X)).

**Example (3-1):** Let X={a,b,c} with the topology \(\tau=\{X, \emptyset, \{a\}\}\) on X , then arPSC(X)= \(\{X, \emptyset, \{b\}, \{c\}, \{b,c\}\}\) and arPSO(X) = \(\{X, \emptyset, \{a\}, \{a,b\}, \{a,c\}\}\).

**Proposition (3-2):** Let (X,\(\tau\)) be a topological space . Then
(i) Every \(\alpha\)-closed set in X is an arps-closed.
(ii) Every closed set in X is an arps-closed.
Every regular closed set in X is a $\alpha$-rps-closed.

**Proof:**
(i) Let $A$ be an $\alpha$-closed set in X and let $A \subseteq U$ where $U$ is a rps-open set in X. Since $A$ is an $\alpha$-closed in X. Thus, we have $\alphacl(A) = A \subseteq U$. Therefore, $\alphacl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is a rps-open set. Hence, $A$ is a $\alpha$-rps-closed set in X.

(ii) Let $A$ be a closed set in X and let $A \subseteq U$ where $U$ is a rps-open set in X. Since $A$ is a closed set in X. Then $cl(A) = A$. But $\alphacl(A) = A \subseteq U$. Thus, we have $\alphacl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is a rps-open set. Hence, $A$ is a $\alpha$-rps-closed set in X.

(iii) It follows from the fact every regular closed set is closed and by, (ii) we have every regular set is a $\alpha$-rps-closed.

**Corollary (3-3):** Let $(X, \tau)$ be a topological space. Then
(i) Every open ($\alpha$-open) set in X is an $\alpha$-rps-open.
(ii) Every regular open set in X is an $\alpha$-rps-open.

**Proof:**
(i) Let $A$ be an open ($\alpha$-open) set in X. Then $A^c$ is a closed ($\alpha$-closed) set in X and by proposition (3-2), (i) and (ii) we get $A^c$ is an $\alpha$-rps-closed in X. Then $A$ is an $\alpha$-rps-open set in X.

(ii) Let $A$ be a regular open set in X. Then $A^c$ is a regular closed set in X and by proposition (3-2), (iii) we get $A^c$ is an $\alpha$-rps-closed in X. Then, $A$ is an $\alpha$-rps-open in X.

**Remark (3-4):** The converse of the proposition (3-2) and corollary (3-3) are not true in general, as the following show.

**Example (3-2):** Let $X=\{a,b,c,d\}$ and $Y=\{a,b,c\}$ be two topological spaces
(i) Consider the topology $\tau = \{X, \Phi, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}\}$. Then the set $\{c\}$ is an $\alpha$-rps-closed set in $(X, \tau)$, but is not closed set in $(X, \tau)$. Also, $\{c\}^c = \{a,b,d\}$ is an $\alpha$-rps-open set in $(X, \tau)$, but is not open set in $(X, \tau)$.

(ii) Consider the topology $\sigma = \{Y, \Phi, \{a\}, \{b\}, \{a, b\}\}$. Then the set $\{c\}$ is an $\alpha$-rps-closed set in $(Y, \sigma)$ but is not regular closed in $(Y, \sigma)$. Also, $\{c\}^c = \{a, b\}$ is an $\alpha$-rps-open set in $(Y, \sigma)$, but is not regular open set in $(Y, \sigma)$.

**Proposition (3-5):** Let $(X, \tau)$ be a topological space. Then
(i) Every $\alpha$-rps-closed set in X is a sg-closed.
(ii) Every $\alpha$-rps-closed set in X is a gs-closed.
(iii) Every $\alpha$-rps-closed set in X is a gsp-closed.
(iv) Every $\alpha$-rps-closed set in X is a gb-closed.

**Proof:**
(i) Let $A$ be an $\alpha$-rps-closed set in X and $A \subseteq U$, where $U$ is a semi-open set in X. By remark (2-5) (Every semi-open set is a rps-open) and since $A$ is an $\alpha$-rps-closed set. Then $\alphacl(A) \subseteq U$. But $\alphacl(A) \subseteq \alphacl(A) \subseteq U$. Thus, we have $\alphacl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is a semi-open set in X. Therefore, $A$ is a sg-closed set in X.

(ii) It follows from (i) and by remark (2-4) (Every sg-closed set is gs-closed). Therefore, every $\alpha$-rps-closed set is a gs-closed.

(iii) It follows from (ii) and by remark (2-4) (Every gs-closed set is gsp-closed). Therefore, every $\alpha$-rps-closed set is a gsp-closed.

(iv) It follows from (ii) and by remark (2-4) (Every gs-closed set is gb-closed). Therefore, every $\alpha$-rps-closed set is a gb-closed.

**Corollary (3-6):** Let $(X, \tau)$ be a topological space. Then
(i) Every $\alpha$-rps-open set in X is a sg-open.
(ii) Every $\alpha$-rps-open set in X is a gs-open.
(iii) Every \( \alpha \text{rps} \)-open set in \( X \) is a \( gsp \)-open.

(iv) Every \( \alpha \text{rps} \)-open set in \( X \) is a \( gb \)-open.

**Proof:** (i) Let \( A \) be an \( \alpha \text{rps} \)-open set in \( X \). Then \( A^c \) is an \( \alpha \text{rps} \)-closed set in \( X \) and by proposition (3-5), (i) we get \( A^c \) is an \( sg \)-closed set in \( X \). Thus, \( A \) is a \( sg \)-open set in \( X \).

(ii) Let \( A \) be an \( \alpha \text{rps} \)-open set in \( X \). Then \( A^c \) is an \( \alpha \text{rps} \)-closed set in \( X \) and by proposition (3-5), (ii) we get \( A^c \) is an \( gs \)-closed set in \( X \). Thus, \( A \) is a \( gs \)-open set in \( X \).

(iii) Let \( A \) be an \( \alpha \text{rps} \)-open set in \( X \). Then \( A^c \) is an \( \alpha \text{rps} \)-closed set in \( X \) and by proposition (3-5), (iii) \( A^c \) is a \( gsp \)-closed set in \( X \). Thus, \( A \) is a \( gsp \)-open set in \( X \).

(iv) Let \( A \) be an \( \alpha \text{rps} \)-open set in \( X \). Then \( A^c \) is an \( \alpha \text{rps} \)-closed set in \( X \) and by proposition (3-5), (iv) we get \( A^c \) is a \( gb \)-closed set in \( X \). Hence, \( A \) is a \( gb \)-open set in \( X \).

**Remark (3-7):** The converse of the proposition (3-5) and corollary (3-6) are not true in general, as the following show:

**Example (3-3):** Let \( X = \{a, b, c, d\} \) with the topology \( \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \). Then the set \( \{a, d\} \) is an \( sg \)-closed (resp. \( gs \)-closed, \( gsp \)-closed and \( gb \)-closed) set in \( (X, \tau) \), but is not \( \alpha \text{rps} \)-closed set in \( (X, \tau) \). Also, \( \{a, d\}^c = \{b, c\} \) is a \( sg \)-open (resp. \( gs \)-open, \( gsp \)-open and \( gb \)-open) set in \( (X, \tau) \), but is not \( \alpha \text{rps} \)-open set in \( (X, \tau) \).

**Proposition (3-8):** Let \( (X, \tau) \) be a topological space. Then

(i) Every \( \alpha \text{rps} \)-closed set in \( X \) is \( g\alpha \)-closed.

(ii) Every \( \alpha \text{rps} \)-closed set in \( X \) is \( \alpha g \)-closed.

(iii) Every \( \alpha \text{rps} \)-closed set in \( X \) is \( gp \)-closed.

(iv) Every \( \alpha \text{rps} \)-closed set in \( X \) is \( gpr \)-closed.

**Proof:** (i) Let \( A \) be an \( \alpha \text{rps} \)-closed set in \( X \). Let \( A \subseteq U \), where \( U \) is an \( \alpha \)-open set in \( X \). By Remark (2-5) and since \( A \) is an \( \alpha \text{rps} \)-closed set in \( X \), then \( \alpha \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) when \( U \) is an \( \alpha \)-open set in \( X \). Hence, \( A \) is a \( g\alpha \)-closed.

(ii) It follows from (i) and by Remark (2-4) (Every \( g\alpha \)-closed set is an \( \alpha g \)-closed). Hence, every \( \alpha \text{rps} \)-closed set is an \( \alpha g \)-closed.

(iii) It follows from (ii) and by Remark (2-4) (Every \( \alpha g \)-closed set is a \( gp \)-closed). Hence, every \( \alpha \text{rps} \)-closed set is a \( gp \)-closed.

(iv) It follows from (iii) and by Remark (2-4) (Every \( gp \)-closed set is a \( gpr \)-closed). Hence, every \( \alpha \text{rps} \)-closed set is a \( gpr \)-closed.

**Corollary (3-9):** Let \( (X, \tau) \) be a topological space. Then

(i) Every \( \alpha \text{rps} \)-closed set is \( rg\alpha \)-closed.

(ii) Every \( \alpha \text{rps} \)-closed set is \( rg \)-closed.

(iii) Every \( \alpha \text{rps} \)-closed set is \( rwg \)-closed.

**Proof:** (i) follows from the proposition (3-8) and Remark (2-4) Therefore, every \( \alpha \text{rps} \)-closed set is an \( rg\alpha \)-closed.

(ii) follows from (i) and Remark (2-4) (Every \( rg\alpha \)-closed set is an \( rg \)-closed). Therefore, every \( \alpha \text{rps} \)-closed set is an \( rg \)-closed (Every \( rg\alpha \)-closed set is an \( rwg \)-closed). Therefore, every \( \alpha \text{rps} \)-closed set is an \( rwg \)-closed.

(iii) follows from (i) and Remark (2-4).

**Corollary (3-10):** Let \( (X, \tau) \) be a topological space. Then

(i) Every \( \alpha \text{rps} \)-open set in \( X \) is a \( g\alpha \)-open set.

(ii) Every \( \alpha \text{rps} \)-open set in \( X \) is an \( \alpha g \)-open set.
(iii) Every $\alpha rps$-open set in $X$ is a gp-open set.
(iv) Every $\alpha rps$-open set in $X$ is a gpr-open set.

**Proof**: (i) Let $A$ be an $\alpha rps$-open set in $X$. Then $A^c$ is an $\alpha rps$-closed set in $X$ and by proposition (3-8), (i) we get $A^c$ is a $g\alpha$-closed in $X$. Hence, $A$ is a $g\alpha$-open set in $X$.

(ii) Let $A$ be a $\alpha rps$-open set in $X$. Then $A^c$ is a $\alpha rps$-closed set in $X$ and by proposition (3-8), (ii) we get $A^c$ is an $ag\alpha$-closed in $X$. Hence, $A$ is an $ag\alpha$-open set in $X$.

(iii) Let $A$ be an $\alpha rps$-open set in $X$. Thus, $A^c$ is an $\alpha rps$-closed set in $X$ and by using proposition (3-8), (iii) we get $A^c$ is a gp-closed set in $X$. Then, $A$ is a gp-open set in $X$.

(iv) Let $A$ be an $\alpha rps$-open set in $X$. Then $A^c$ is a $\alpha rps$-closed set in $X$ and by proposition (3-8), (IV) we get $A^c$ is a gpr-closed set in $X$. Hence, $A$ is a gpr-open set in $X$.

The proof of the following corollary it is easy. Hence, it is omitted.

**Corollary (3-11)**: Let $(X, \tau)$ be a topological space. Then

(i) Every $\alpha rps$-open set is $rg\alpha$-open.
(ii) Every $\alpha rps$-open set is $rg$-open.
(iii) Every $\alpha rps$-open set is $rwg$-open.

**Remark (3-12)**: The converse of the proposition (3-8) and corollary (3-9), (3-10) and corollary (3-11) need not be true in general, as seen from the following example:

**Example (3-4)**: Let $X = \{a, b, c, d\}$ and $Y = \{a, b, c\}$ be two topological spaces

(i) Consider the topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then the set $\{b, d\}$ is an $ag\alpha$-closed (resp. gp-closed and gpr-closed) set in $(X, \tau)$, but is not $\alpha rps$-closed set in $(X, \tau)$. Also, $\{b, d\}^c = \{a, c\}$ is an $ag\alpha$-open (resp. gp-open and gpr-open) set in $(X, \tau)$, but is not $\alpha rps$-open set in $(X, \tau)$.

(ii) Consider the topology $\sigma = \{Y, \emptyset, \{a\}, \{b, c\}\}$. Then the set $\{c\}$ is a $ga\alpha$-closed set (resp. $rg\alpha$-closed, rg-closed and $rwg$-closed) in $(Y, \sigma)$ but is not $\alpha rps$-closed set in $(Y, \sigma)$. Also, $\{c\}^c = \{a, b\}$ is an $ga\alpha$-open (resp. $rg\alpha$-open, rg-open and $rwg$-open) set in $(Y, \sigma)$ but is not $\alpha rps$-open set in $(Y, \sigma)$.

**Remark (3-13)**: The concept of $g\alpha$-closed set and $\alpha rps$-closed set are independent.

**Example (3-5)**: Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then $\{a, d\}$ is a $g\alpha$-closed set in $(X, \tau)$. But is not $\alpha rps$-closed set in $(X, \tau)$ and $\{c\}$ is an $\alpha rps$-closed set in $(X, \tau)$ but is not $g\alpha$-closed set in $(X, \tau)$.

The following propositions given the condition to make $g\alpha$-closed sets and $\alpha rps$-closed sets are equivalent.

**Proposition (3-14)**: If $X$ is a $T_{1/2}$-space, and then every $g\alpha$-closed set in $X$ is $\alpha rps$-closed.

**Proof**: Let $A$ be a $g\alpha$-closed set in $X$. Since $X$ is a $T_{1/2}$-space and by definition (2-6) we get $A$ is a closed in $X$ and by proposition (3-2), (ii), $A$ is an $\alpha rps$-closed set in $X$.

**Proposition (3-15)**: If $X$ is a $T_b$-space, and then every $\alpha rps$-closed set in $X$ is $g\alpha$-closed.

**Proof**: Let $A$ be a $\alpha rps$-closed set in $X$ and by corollary (3-9), (ii) we get $A$ is a $g\alpha$-closed in $X$. Since $X$ is a $T_b$-space and by definition (2-6), (2) we get $A$ is a closed
set in X and since every closed set is a g-closed set Therefore, A is a g-closed set in X.

In following proposition and next results, we introduce some properties of $\alpha_{rps}$-closed sets:

**Proposition (3-16):** A subset A of a space X is an $\alpha_{rps}$-closed set if and only if $\alpha_{cl}(A) - A$ does not contain any non-empty rps-closed set in X.

**Proof:** Let $A$ be a $\alpha_{rps}$-closed set in X. We prove the result by contradiction. Let $G \neq \emptyset$ be an rps-closed set in X such that $G \subseteq \alpha_{cl}(A) - A$, since $\alpha_{cl}(A) - A = \alpha_{cl}(A) \cap A^c$. Therefore, $G \subseteq \alpha_{cl}(A)$ and $G \subseteq A^c$. Since $X - G$ is an rps-open set in X and $A$ is an $\alpha_{rps}$-closed set, thus $\alpha_{cl}(A) \subseteq X - G$. That is $G \subseteq [\alpha_{cl}(A)]^c$. Hence, $G \subseteq \alpha_{cl}(A) \cap [\alpha_{cl}(A)]^c = \emptyset$. That is $G = \emptyset$ (which is contradiction). Since $G \neq \emptyset$. Thus, $\alpha_{cl}(A) - A$ does not contain any non-empty rps-closed set in X. Conversely, suppose $\alpha_{cl}(A) - A$ does not contain any non-empty rps-closed set. Since $\alpha_{cl}(A) - A$ is a closed subset, then $\alpha_{cl}(A) - A = \emptyset$ Thus, $\alpha_{cl}(A) = A$ and so $A$ is an $\alpha_{rps}$-closed set in X.

**Corollary (3-17):** If $A$ is an $\alpha_{rps}$-closed set in X. Then $A$ is an $\alpha$-closed set if and only if $\alpha_{cl}(A) - A$ is closed.

**Proof:** Let $A$ be an $\alpha_{rps}$-closed set in X, if $A$ is an $\alpha$-closed set, then we get $\alpha_{cl}(A) = A$. Thus, $\alpha_{cl}(A) - A$ which is closed set. Conversely, let $\alpha_{cl}(A) - A$ be a closed set. Then by proposition (3-16) we have $\alpha_{cl}(A) - A$ does not contain any non-empty rps-closed set. Since $\alpha_{cl}(A) - A$ is a closed subset, then $\alpha_{cl}(A) - A = \emptyset$ Thus, $\alpha_{cl}(A) = A$ and so $A$ is an $\alpha$-closed set.

**Proposition (3-18):** The union of two $\alpha_{rps}$-closed subsets of X is also $\alpha_{rps}$-closed set in X.

**Proof:** Let $A$ and $B$ be two $\alpha_{rps}$-closed set in X. Let $U$ be an rps-open set in X, such that $A \cup B \subseteq U$. Since $A$ and $B$ are $\alpha_{rps}$-closed set in X. Hence, $\alpha_{cl}(A) \subseteq U$ and $\alpha_{cl}(B) \subseteq U$. Hence, $\alpha_{cl}(A \cup B) = (\alpha_{cl}(A) \cup \alpha_{cl}(B)) \subseteq U$. That is $\alpha_{cl}(A \cup B) \subseteq U$ whenever $U$ is a rps-open in X. Then $A \cup B$ is an $\alpha_{rps}$-closed set in X.

**Proposition (3-19):** If $A$ is an $\alpha_{rps}$-closed subset in X and $A \subseteq B \subseteq \alpha_{cl}(A)$ Then B is an $\alpha_{rps}$-closed set in X.

**Proof:** Let A be an $\alpha_{rps}$-closed subset in X, such that $A \subseteq B \subseteq \alpha_{cl}(A)$. Since A is an $\alpha_{rps}$-closed set in X, we get $\alpha_{cl}(A) \subseteq G$. Now, $\alpha_{cl}(A) \subseteq \alpha_{cl}(B) \subseteq \alpha_{cl}(\alpha_{cl}(A)) = \alpha_{cl}(B) \subseteq G$. Hence, G is an rps-open set in X. The converse need not be true in general. As seen from the following example:

**Example (3-6):** Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{a, c\}\}$ on X. Let $A = \{b\}$ and $B = \{b, c\}$ be two $\alpha_{rps}$-closed sets in $(X, \tau)$ such that $A \subseteq B$, But $B \not\subseteq \alpha_{cl}(A)$.

**Proposition (3-20):** Let A and B be two rps-open subsets of X. Then $A \cap B$ is an $\alpha_{rps}$-closed set in X.

**Proof:** If A and B be two rps-open subsets of X. Then $A^c$ and $B^c$ are both $\alpha_{rps}$-closed set in X. By proposition (3-18) we get $A^c \cup B^c$ is also $\alpha_{rps}$-closed set in X. But $A^c \cup B^c = (A \cap B)^c$ is rps-closed set in X. Thus, $A \cap B$ is $\alpha_{rps}$-closed in X.

**Proposition (3-21):** Let A be any subset of a topological space $(X, \tau)$, if A is an rps-open and $\alpha_{rps}$-closed set in X. Then A is $\alpha$-closed set.
Proof: Suppose that a subset A of a space X is an rps-open and arps-closed set in X. Thus, A \subseteq \text{acl}(A) and \text{acl}(A) \subseteq A. Then, A = \text{acl}(A). Hence, A is an \alpha-closed set in X.

Corollary (3.22): If A is an rps-open and arps-closed subset in X and F is an \alpha-closed set in X. Then A \cap F is an arps-closed set in X.

Proof: Suppose that A be an rps-open and arps-closed subset in X. Then by proposition (3.21) we get A is an \alpha-closed set in X. Since F is an \alpha-closed set in X then by Remark(2-2), \text{iii} we have A \cap F is an \alpha-closed set in X. Also, by proposition (3.2), (i) we obtain \text{A \cap F} is an arps-closed set in X.

Proposition (3.23): For an element x \in X, the set X \setminus \{x\} is a arps-closed set or rps-open set.

Proof: Let X \setminus \{x\} is not rps-open set. Then X is the only rps-open set containing X \setminus \{x\}. This implies \text{acl}(X \setminus \{x\}) \subseteq X. Therefore, X \setminus \{x\} is arps-closed set in X.

Next, we introduce Tαrps-space as an application of arps-closed sets in topological spaces and we study some of it properties.

Definition (3.24): A topological space (X,τ) is said to be Tαrps-space if every arps-closed set in X is an \alpha-closed set.

Proposition (3.25): A space (X,τ) is a Tαrps-space if and only if every singleton set in X is rps-closed or \alpha-open.

Proof: Let X be a Tαrps-space. To prove every singleton set in X is rps-closed or \alpha-open. Suppose \{x\} is not rps-closed set. Then X \setminus \{x\} is not rps-open set. Thus, X is the only rps-open set containing X \setminus \{x\} and hence \text{acl}(X \setminus \{x\}) \subseteq X. Then X \setminus \{x\} is arps-closed in X. Since X is a Tαrps-space and by definition (3.24) we have X \setminus \{x\} is an \alpha-closed set in X. Therefore, \{x\} is an \alpha-open set.

Conversely, Let A is a arps-closed set in X. Since A is an arps-closed set in X and by proposition (3.16) we get \text{acl}(A) - A does not contain any non-empty arps-closed set in X. Let x \in \text{acl}(A). By our assumption \{x\} is either rps-closed set or \alpha-open set. Case (i): Let x \in \text{acl}(A) such that \{x\} is an rps-closed. Since \{x\} is an rps-closed set and by proposition (3.16) we get x \notin \text{acl}(A) - A. Thus, x \notin A Therefore, A = \text{acl}(A). Hence, A is an \alpha-closed set. Case (ii): Let \{x\} be not rps-closed for each x \in \text{acl}(A). Now if x \in \text{acl}(A). Then \{x\} is an \alpha-open set and \{x\} \cap A \neq \emptyset that implies x \notin A. Therefore, A = \text{acl}(A). Hence, A is an \alpha-closed set. From case (i) and case (ii) it follows from definition (3.24) we obtain X is a Tαrps-space.

Proposition (3.26): If (X,τ) is a Tαrps-space then every arps-open set in X is an \alpha-open set.

Proof: Let A be an arps-open set in X. Then A^c is an arps-closed set in X. Since X is a Tαrps-space and by definition (3.24) we get A^c is an \alpha-closed set in X Hence, A is an \alpha-open set in X.

Proposition (3.27): Every T_{1/2} space is a Tαrps-space.

Proof: Let (X,τ) be a T_{1/2} space and let A be an arps-closed set in X. Then by corollary (3.9), (ii) we get A is an rg-closed set in X. Since X is a T_{1/2} space and by definition (2-6), (ii) we have A is an \alpha-closed set in X and by Remark(2-2), (ii) we obtain A is an \alpha-closed set in X. Therefore, X is a Tαrps-space.

The converse of proposition (3.27) need not be true in general. As seen from the following example:
Example (3-7): Let $X=\{a,b,c\}$ with the topology $\tau=\{X,\phi,\{a\},\{a,c\}\}$. Since $\alpha\text{RPSC}(X)=\{X,\phi,\{b\},\{c\}\}$ are $\alpha$-closed sets in $(X,\tau)$. Then $X$ is a $T_{\alpha\text{arps}}$-space. But it is not $T_{*1/2}$-space. Since, the set of all rg-closed in $(X,\tau)$ are $\{X,\phi,\{a\},\{b\},\{c\}\}$ are $\alpha$-closed sets in $(X,\tau)$.

Proposition (3-28): Every $T_b$-space is a $T_{\alpha\text{arps}}$-space.

Proof: Let $(X,\tau)$ be a $T_b$-space and let $A$ be an $\alpha\text{arps}$-closed set in $X$. By proposition (3-5), (ii) we get $A$ is a $gs$-closed set in $X$. Since $X$ is a $T_b$-space and by definition (2-6), (ii) we have $A$ is a closed set in $X$ and also by Remark (2-2), (ii). We get $A$ is an $\alpha$-closed set in $X$. Hence, $X$ is a $T_{\alpha\text{arps}}$-space.

The converse of proposition (3-28) need not be true in general. As seen from the following example:

Example (3-8): Let $X=\{a,b,c\}$ with the topology $\tau=\{X,\phi,\{a\},\{b\}\}$. Since $\alpha\text{RPSC}(X)=\{X,\phi,\{a\},\{b\}\}$ are $\alpha$-closed sets in $(X,\tau)$. Then $X$ is a $T_{\alpha\text{arps}}$-space. But it is not $T_{b}$-space. Since, the set of all $gs$-closed in $(X,\tau)$ are $\{X,\phi,\{a\},\{b\}\}$ are $\alpha$-closed sets in $(X,\tau)$.

Proposition (3-29): Every $\alpha T_b$-space is a $T_{\alpha\text{arps}}$-space.

Proof: Let $(X,\tau)$ be an $\alpha T_b$-space and let $A$ be an $\alpha\text{arps}$-closed set in $X$. By using proposition (3-8), (ii) we get $A$ is an $\alpha g$-closed set in $X$. Since $X$ is an $\alpha T_b$-space and by definition (2-6), (ii) we have $A$ is a closed set in $X$ and also by Remark (2-2), (ii). Thus, $A$ is an $\alpha$-closed set in $X$. Hence, $X$ is a $T_{\alpha\text{arps}}$-space.

The converse of proposition (3-29) need not be true in general. As seen from the following example:

Example (3-9): Let $X=\{a,b,c\}$ with the topology $\tau=\{X,\phi,\{a\},\{b\}\}$. Since $\alpha\text{RPSC}(X)=\{X,\phi,\{a\},\{b\}\}$ are $\alpha$-closed sets in $(X,\tau)$. Then $X$ is a $T_{\alpha\text{arps}}$-space. But it is not $\alpha T_b$-space. Since, the set of all $gs$-closed in $(X,\tau)$ are $\{X,\phi,\{a\},\{b\}\}$ are $\alpha$-closed sets in $(X,\tau)$.

Proposition (3-30): If $X$ is a $T_{*1/2}$-space, then every $\alpha\text{arps}$-closed set in $X$ is closed.

Proof: Let $A$ be an $\alpha\text{arps}$-closed set in $X$. By corollary (3-9), (ii) we get $A$ is an rg-closed in $X$. Since $X$ is a $T_{*1/2}$-space, then, $A$ is a closed set in $X$.

The following proposition and corollary is easy. Thus, we omitted the proofs:

Proposition (3-31): Let $(X,\tau)$ be a topological space. Then every $\alpha\text{arps}$-closed set in $X$ is closed if

(i) $X$ is $T_{b}$-space
(ii) $X$ is $\alpha T_b$-space.

Corollary (3-32): Let $(X,\tau)$ be a topological space. Then every $\alpha\text{arps}$-open set in $X$ is open if $X$ is

(i) $T_{*1/2}$-space
(ii) $\alpha T_b$-space, (iii) $T_b$-space.

Remark (3-33): The converse of proposition (3-30), (3-31) and corollary (3-32) need not be true in general. It is easy seen that in example (3-9) (Every $\alpha\text{arps}$-closed set in $(X,\tau)$ is closed and also every $\alpha\text{arps}$-open set in $(X,\tau)$ is open). But $X$ is not $T_b$-space, not $\alpha T_b$-space and not $T_{*1/2}$-space.
ARPS-CONTINUOUS FUNCTION, αRPS-IRRESOLUTE FUNCTION and STRONGLY αRPS-CONTINUOUS FUNCTION.

In this section, we introduce and study new types of continuous functions and will be study some of their properties.

**Definition (4-1):** A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be arps-continuous continuous if \( f^{-1}(A) \) is an arps-closed set in \( X \) for every closed set \( A \) in \( Y \).

**Proposition (4-2):** A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is arps- continuous if and only if \( f^{-1}(A) \) is an αrps-open set in \( X \) for every open set \( A \) in \( Y \).

**Proof:** Let \( f \) be an αrps-continuous function and \( A \) be an open set in \( Y \). Then \( A^c \) is a closed set in \( Y \). Thus, \( f^{-1}(A^c) \) is an αrps-closed set in \( X \). By assumption \( f^{-1}(A^c) \) is an αrps-closed set in \( X \). Hence, \( f^{-1}(A) \) is an αrps-open set in \( X \). Conversely, let \( A \) be a closed in \( Y \). Then \( A^c \) is an open set in \( Y \). By corollary (3-32) we get \( f^{-1}(A) \) is an αrps-closed set in \( X \).

**Proposition (4-3):** Every continuous function is αrps-continuous.

**Proof:** Follows from the definition (4-1) and the fact every closed set is an αrps-closed. The converse need not be true in general. As see from the following example:

**Example (4-1):** Let \( X = \{a,b,c\} \), \( \tau = \{X,\emptyset,\{a\}\} \) and \( \sigma = \{Y,\emptyset,\{a,b\}\} \) on \( X,Y \) respectively. Let \( f : (X,\tau) \rightarrow (Y,\sigma) \) be the identity function. Then \( f \) is an arps-continuous function, but is not continuous. Since for the closed set \{c\} in \( Y \), \( f^{-1}(\{c\}) = \{c\} \) is not closed set in \( X \).

**Proposition (4-4):** Let \( f : (X,\tau) \rightarrow (Y,\sigma) \) be a function from topological space \( X \) in to topological space \( Y \). If \( f \) is an αrps-continuous function, then \( f \) is a:

(i) gα-continuous function.
(ii) αg-continuous function.
(iii) gp-continuous function.
(iv) gpr-continuous function.
(v) rgα-continuous function.
(vi) rg-continuous function.

**Proof:** (i) Let \( A \) be closed set in \( Y \). Then, \( f^{-1}(A) \) is an αrps-closed set in \( X \). By proposition (3-8) and corollary (3-9), we get the following proposition and it is prove easy. Thus, we omitted it is.

**Proposition (4-6):** Every αrps-continuous function, \( f : (X, \tau) \rightarrow (Y, \sigma) \) is a:

(i) ga-continuous.
(ii) a-continuous.
(iii) gp-continuous.
(iv) gpr-continuous.
(v) g-continuous.
(vi) rg-continuous.
(vii) rwg-continuous.

The converse of a proposition (4-5) and (4-6) may not be true in general, as shown in the following example.

Example (4-2): Let \( X = Y = \{a, b, c\} \) with the topologies \( \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \) and \( \sigma = \{Y, \emptyset, \{a\}, \{b, c\}\} \) on \( X \) and \( Y \) respectively. Let \( f : (X, \tau) \to (Y, \sigma) \) be the identity function. Then it is observe that \( f \) is not \( \alpha \text{rps-continuous} \) since for the closed set \( \{a\} \) in \( (Y, \sigma) \), \( f^{-1}(\{a\}) = \{a\} \) is not \( \alpha \text{rps-closed} \) in \( (X, \tau) \). However, \( f \) is sg-continuous (resp. gs-continuous, gsp-continuous, gb-continuous) functions.

Example (4-3): Let \( X = Y = \{a, b, c\} \) with the topologies \( \tau = \{X, \emptyset, \{a\}, \{b, c\}\} \) and \( \sigma = \{Y, \emptyset, \{a\}, \{a, c\}\} \) on \( X \) and \( Y \) respectively. Let \( f : (X, \tau) \to (Y, \sigma) \) be the identity function. Then \( f \) is not \( \alpha \text{rps-continuous} \). Since for the closed set \( \{b\} \) in \( (Y, \sigma) \), \( f^{-1}(\{b\}) = \{b\} \) is not \( \alpha \text{rps-closed} \) in \( (X, \tau) \). However, \( f \) is a \( g\alpha \text{-continuous} \) (\( \alpha \text{g-continuous}, \text{gp-continuous}, \text{gpr-continuous}, \text{rg}\alpha \text{-continuous}, \text{rg-continuous and rwg-continuous}) functions.

Proposition (4-7): If \( f : (X, \tau) \to (Y, \sigma) \) is any function from topological space \( X \) in to topological space \( Y \) and \( X \) is a \( T_{b} \)-space

(i) If \( f \) is a sg-continuous function, then it is \( \alpha \text{rps-continuous} \).

(ii) If \( f \) is a gs-continuous function, then it is \( \alpha \text{rps-continuous} \).

Proof: (i) Let \( f : (X, \tau) \to (Y, \sigma) \) be a sg-continuous function and let \( A \) be a closed set in \( Y \). Thus, \( f^{-1}(A) \) is a sg-closed set in \( X \), since (Every sg-closed set is gs-closed). Then \( f^{-1}(A) \) is gs-closed in \( X \) and since \( X \) is a \( T_{b} \)-space, then \( f^{-1}(A) \) is a closed set in \( X \), and by proposition (3-2), (ii) we get \( f^{-1}(A) \) is an \( \alpha \text{rps-closed} \) set in \( X \). Therefore, \( f \) is an \( \alpha \text{rps-continuous} \).

(ii) Let \( f : (X, \tau) \to (Y, \sigma) \) be a gs-continuous function and \( A \) be a closed set in \( Y \). Thus, \( f^{-1}(A) \) is an gs-closed set in \( X \) and since \( X \) is a \( T_{b} \)-space. Thus, \( f^{-1}(A) \) is a closed set in \( X \), and by using proposition (3-2), (ii) we get \( f^{-1}(A) \) is an \( \alpha \text{rps-closed} \) set in \( X \). Therefore, \( f \) is a \( \alpha \text{rps-continuous} \).

Similarly, we prove the following proposition:

Proposition (4-8): If \( f : (X, \tau) \to (Y, \sigma) \) is any function and let \( X \) be an \( \alpha T_{b} \)-space and \( T^{*}_{1/2} \)-space.

(i) If \( f \) is an \( \alpha \text{g-continuous function} \) , then it is \( \alpha \text{rps-continuous} \).

(ii) If \( f \) is a \( \alpha \text{g-continuous function} \), then it is \( \alpha \text{rps-continuous} \).

(iii) If \( f \) is rg-continuous function, then it is \( \alpha \text{rps-continuous} \).

(iv) If \( f \) is \( \alpha \text{rga-continuous function} \), then it is \( \alpha \text{rps-continuous} \).

Now, we given other type of \( \alpha \text{rps-continuous} \) function is called \( \alpha \text{rps-irresolute} \):

Definition (4-9): A function \( f : (X, \tau) \to (Y, \sigma) \) is said to be \( \alpha \text{rps-irresolute} \) if \( f^{-1}(A) \) is an \( \alpha \text{rps-closed} \) set in \( X \) for every \( \alpha \text{rps-closed} \) set \( A \) in \( Y \).

Proposition (4-10): A function \( f : (X, \tau) \to (Y, \sigma) \) is \( \alpha \text{rps-irresolute} \) if and only if \( f^{-1}(A) \) is an \( \alpha \text{rps-open} \) set in \( X \) for every \( \alpha \text{rps-open} \) set \( A \) in \( Y \).

Proof: This proof is similar to that of proposition (4-2).

Proposition (4-11): Every \( \alpha \text{rps-irresolute} \) function is \( \alpha \text{rps-continuous} \).

Proof: Let \( f : (X, \tau) \to (Y, \sigma) \) be an \( \alpha \text{rps-irresolute} \) function and \( A \) be a closed set in \( Y \). By proposition (3-2), (ii) we have \( A \) is an \( \alpha \text{rps-closed} \) set in \( Y \). Thus, \( f^{-1}(A) \) is an \( \alpha \text{rps-closed} \) in \( X \). Therefore, \( f \) is an \( \alpha \text{rps-irresolute} \) function. The converse need not be true.
Example (4-4): Let \( X=Y=\{a,b,c,d\} \) with the topologies \( \tau=\{X, \emptyset, \{a\}, \{b\}, \{a,b\} \}, \sigma=\{Y, \Phi, \{d\}, \{c,d\}, \{a,c,d\}, \{b,c,d\}\} \) on \( X \) and \( Y \) respectively.

Define \( f:(X,\tau)\rightarrow(Y,\sigma) \) by \( f(a)=c, f(b)=f(c)=b \) and \( f(d)=d \). It is observed that \( f \) is a \( \alpha_rps \)-continuous, but is not \( \alpha_rps \)-irresolute. Since for the \( \alpha_rps \)-closed set \( \{c\} \) in \((Y,\sigma), f^{-1}(\{c\})=\{a\} \) is not an \( \alpha_rps \)-closed set in \((X,\tau)\).

Remark (4-12): The notions \( \alpha_rps \)-irresolute function and continuous function are independent. It is observed seen that in Example (4-1), such that \( f \) is an \( \alpha_rps \)-continuous function. But \( f \) is not a \( \alpha_rps \)-irresolute function. Since for the closed set \( \{c\} \) in \((X,\tau), f^{-1}(\{c\})=\{c\} \) is not a closed set in \((X,\tau)\). Also in Example (4-4) it is easy seen that \( f \) is a \( \alpha_rps \)-continuous but is not an \( \alpha_rps \)-irresolute function.

Proposition (4-13): If \( f:(X,\tau)\rightarrow(Y,\sigma) \) is an \( \alpha_rps \)-continuous function and \( Y \) is a \( T_{b^*} \)-space. Then \( f \) is an \( \alpha_rps \)-irresolute function.

Proof: Let \( A \) be an \( \alpha_rps \)-closed set in \( Y \). By proposition (3-5), (ii) we get \( A \) is a \( g \)-closed set in \( Y \). Since \( Y \) is a \( T_{b^*} \)-space, then \( A \) is a closed set in \( Y \), and so, \( f^{-1}(A) \) is an \( \alpha_rps \)-closed set in \( X \). Hence, \( f \) is an \( \alpha_rps \)-irresolute function.

The following proposition it is easy. Thus, we omitted the proof:

Proposition (4-14):
1. If \( f:(X,\tau)\rightarrow(Y,\sigma) \) is an \( \alpha_rps \)-irresolute function and \( X \) is a \( T_{b^*} \)-space, then \( f \) is a \( \alpha_rps \)-continuous function.
2. If \( f:(X,\tau)\rightarrow(Y,\sigma) \) is a continuous function and \( Y \) is a \( T_{b^*} \)-space, then \( f \) is an \( \alpha_rps \)-irresolute function.

Proposition (4-15): If \( f:(X,\tau)\rightarrow(Y,\sigma) \) and \( g:(Y,\sigma)\rightarrow(Z,\mu) \) are both \( \alpha_rps \)-irresolute. Then \( g\circ f:(X,\tau)\rightarrow(Z,\mu) \) is an \( \alpha_rps \)-irresolute function.

Proof: Let \( A \) be an \( \alpha_rps \)-closed set in \( Z \). Thus \( g^{-1}(A) \) is an \( \alpha_rps \)-closed set in \( Y \). Since \( f \) is an \( \alpha_rps \)-continuous function, then \( f^{-1}(g^{-1}(A)) \) is an \( \alpha_rps \)-closed set in \( X \). But \( f^{-1}(g^{-1}(A)) = (g\circ f)^{-1}(A) \). Hence, \( g\circ f:(X,\tau)\rightarrow(Z,\mu) \) is an \( \alpha_rps \)-irresolute function.

Similarly, we prove the following proposition:

Proposition (4-16): If \( f:(X,\tau)\rightarrow(Y,\sigma) \) and \( g:(Y,\sigma)\rightarrow(Z,\mu) \) are any two functions. Then \( g\circ f:(X,\tau)\rightarrow(Z,\mu) \) is an \( \alpha_rps \)-continuous function if:

(i) \( f \) is \( \alpha_rps \)-irresolute and \( g \) is \( \alpha_rps \)-continuous.
(ii) \( f \) is \( \alpha_rps \)-continuous and \( g \) is continuous.

Remark (4-17): If \( f:(X,\tau)\rightarrow(Y,\sigma) \) and \( g:(Y,\sigma)\rightarrow(Z,\mu) \) are both \( \alpha_rps \)-continuous function. Then \( g\circ f:(X,\tau)\rightarrow(Z,\mu) \) is not necessarily \( \alpha_rps \)-continuous function. It is easy seen that from the following example:

Example (4-5): Let \( X=Y=Z=\{a,b,c\} \) with topologies \( \tau=\{X, \emptyset, \{a\}, \{b\}\}, \sigma=\{Y, \Phi, \{d\}, \{c,d\}\} \) on \( X, Y \) and \( Z \) respectively. Let \( f:(X,\tau)\rightarrow(Y,\sigma) \) be the identity function and defined \( g:(Y,\sigma)\rightarrow(Z,\mu) \) by \( g(a)=a, g(b)=c \) and \( g(c)=b \). Then \( f \) and \( g \) are \( \alpha_rps \)-continuous. But \( g\circ f \) is not \( \alpha_rps \)-continuous function. Since for the the \( \alpha_rps \)-closed set \( \{b\} \) in \((Z,\mu)\). \((g\circ f)^{-1}(\{b\})=\{c\} \) is not an \( \alpha_rps \)-closed set in \((X,\tau)\).

Proposition (4-18): If \( f:(X,\tau)\rightarrow(Y,\sigma) \) and \( g:(Y,\sigma)\rightarrow(Z,\mu) \) are both \( \alpha_rps \)-continuous function and \( Y \) is a \( T_{1/2} \)-space. Then their composition \( g\circ f:(X,\tau)\rightarrow(Z,\mu) \) is also \( \alpha_rps \)-continuous function.

Proof: It is clear.
In the following definition, we introduce another type of arps-continuous function.

**Definition (4-19):** A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is called strongly arps-continuous if \( f^{-1}(A) \) is closed set in \( X \) for every arps- closed set \( A \) in \( Y \).

**Proposition (4-20):** If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is a strongly arps-continuous if and only if \( f^{-1}(A) \) is open set in \( X \) for every arps- open set \( A \) in \( Y \).

**Proof:** This proof is similar to that of proposition (4-2).

**Proposition (4-21):** Every strongly arps-continuous function is continuous.

**Proof:** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a strongly arps-continuous function and \( A \) be a closed set in \( Y \). By proposition (3-2) , (ii) we have \( A \) is an arps-closed set in \( Y \) . Thus, \( f^{-1}(A) \) is a closed set in \( X \). Therefore, \( f \) is continuous .The converse need not be true in general . As seen from the following example:

**Example (4-6):** Let \( X=\{a,b,c,d\} \) and \( \tau=\{X,\Phi,\{a\},\{a,c\}\} \) on \( X \). Define \( f : (X, \tau) \rightarrow (X, \tau) \) by \( f(a)=a, f(b)=b \) and \( f(c)=c \). Then \( f \) is continuous , but is not strongly arps-continuous . Since for the closed set \( \{c\} \) in \( (X, \tau) \) , \( f^{-1}(\{c\})=\{a\} \) is not a closed in \( (X, \tau) \).

**Proposition (4-22):** Let \( (X, \tau) \) be any topological space, \( Y \) be a T\(_{1/2}\) - space and \( f : (X, \tau) \rightarrow (Y, \sigma) \) is a any function. Then the following are equivalent:

(i) \( f \) is a strongly arps-continuous function .

(ii) \( f \) is a continuous function .

**Proof:** (i) \( \rightarrow \) (ii) Follows from proposition (4-21).

(ii) \( \rightarrow \) (i) Let \( A \) be an arps-closed set in \( Y \). Since \( Y \) is a T\(_{1/2}\) - space and by proposition (3-30) we get \( A \) is a closed set in \( Y \), then \( f^{-1}(A) \) is a closed set in \( X \). Therefore, \( f \) is an arps-continuous function.

**Corollary (4-23):** Every strongly arps-continuous function is an arps-continuous function.

**Remark (4-25):** The converse of a corollary (4-24) need not be true in general. In example (4-7), \( f \) is a arps-irresolute, but is not strongly arps-continuous function.

**Proposition (4-26):** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be any function and \( X,Y \) are both T\(_{1/2}\) - spaces . Then the following are equivalent :

(i) \( f \) is a strongly arps-continuous function .

(ii) \( f \) is a continuous function .

(iii) \( f \) is an arps-irresolute function .

(iv) \( f \) is an arps-continuous function .

**Proof:** Follows from a propositions (4-21), (4-22) and corollaries (4-23), (4-24).

**Proposition (4-27):** If \( f : (X, \tau) \rightarrow (Y, \sigma) \) and \( g : (Y, \sigma) \rightarrow (Z, \mu) \) are both strongly arps-continuous function, then \( g \circ f : (X, \tau) \rightarrow (Z, \mu) \) is also strongly arps-continuous.
**Proof:** Let $A$ be an αrps-closed set in $Z$. Thus, $g^{-1}(A)$ is a closed set in $Y$. By proposition (3-2), ii-we get $g^{-1}(A)$ is an αrps-closed set in $Y$, since $f$ is a strongly αrps-continuous, then $f^{-1}(g^{-1}(A))$ is a closed set in $X$. But $f^{-1}(g^{-1}(A))=(g0f)^{-1}(A)$. Therefore, $g0f:(X, \tau) \rightarrow (Z, \mu)$ is a strongly αrps-continuous function.

Similarly, we prove the following propositions.

**Proposition (4-28):** If $f:(X, \tau) \rightarrow (Y, \sigma)$ is a continuous function and $g:(Y, \sigma) \rightarrow (Z, \mu)$ is strongly αrps-continuous, $g0f:(X, \tau) \rightarrow (Z, \mu)$ is strongly αrps-continuous.

**Proposition (4-29):** If $f:(X, \tau) \rightarrow (Y, \sigma)$ is a αrps-continuous function (or αrps-irresolute) and $g:(Y, \sigma) \rightarrow (Z, \mu)$ is strongly αrps-continuous function Then $g0f:(X, \tau) \rightarrow (Z, \mu)$ is an αrps-irresolute function.

**REFERENCES**


