On $g^s$-Closed Functions in Topological Spaces

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ABSTRACT

Pushpalatha and K. Anitha [1] introduce a new concept of functions which is $g^s$-closed function in topological spaces. In this paper, we modify this concept and study the properties of some types of these functions.

Keywords: $((g^s$ -closed set, $g^s$ –closed function, strongly $g^s$ –closed function irresolve $g^s$-closed function))

INTRODUCTION

B. Navalagi in [3] introduce new class of closed sets is said to be $g^s$-closed set in topological spaces. In 2011 Pushpalatha and K. Anitha [1], are studying the properties of these class of closed sets and investigated new types of continuous functions which are $g^s$-continuous functions upon $g^s$-closed sets with give some properties and relationships of these functions as well as $g^s$-closed functions.

In this work, we modify the definition of $g^s$-closed function appear in [1] by give another types of these functions with study the relationship of these types and some properties.

Definition (1.1),[2]:-
Let $(X, T)$ be a Topological space a subset $A$ of $X$ is said to be semi closed set if $A \subseteq \text{cl}(\text{int}(A))$. The complement of semi closed is said to be semi open.

Definition (1.2),[2]:-
Let $(X, T)$ be a Topological space. The semi closure of $A$ is denoted by $\text{Scl}(A)$ and defined the intersection of all semi closed sets containing $A$.

Definitions (1.3):-
Let $(X, T)$ be a Topological space, a subset $A$ of $X$ is said to be:

1) generalized closed [4 ] (briefly, g-closed) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$, where $U$ is open set in
The complement of g-closed set is g-open set.

(2) generalized semi closed [5] (briefly, gs-closed) if $s\, cl(A) \subseteq U$ whenever $A \subseteq U$, where $U$ is open set in $X$. The complement of gs-closed set is gs-open set.

(3) semi generalized closed [3] (briefly, sg-closed) if $s\, cl(A) \subseteq U$ whenever $A \subseteq U$, where $U$ is semi open set in $X$. The complement of sg-closed set is sg-open set.

(4) s* g-closed [5], if $cl(A) \subseteq U$ whenever $A \subseteq U$, where $U$ is semi open set in $X$. The complement of s* g-closed set is s* g-open set.

(5) g*s-closed [1], if $scl(A) \subseteq U$ whenever $A \subseteq U$, where $U$ is gs- open set in $X$. The complement of g*s-closed set is g*s-open set.

Remarks (1.4),[1],[5]:-
We summarize the fundamental relationships between several types of generalized closed set in the following diagram.

$$
\text{closed} \quad \longrightarrow \quad s^* \text{g-closed} \quad \longrightarrow \quad \text{g-closed} \\
\downarrow \quad \downarrow \\
\text{sg-closed} \quad \longleftarrow \quad g^* \text{s-closed} \quad \longrightarrow \quad \text{gs-closed}
$$

all examples of this diagram can be seen in [1], [5].

Some Types Of g*s-Closed Functions:
In this section, we introduce some types of g*s-closed functions which are strongly g*s-closed functions and irresolute g*s-closed functions with some properties of these types of functions and shows the relationships between these types of functions. These types are a modification of definition appears in [1].

Definition (2.1), [1]:-

$f : (X, T) \longrightarrow (Y, \sigma)$ Let $(X, T)$ and $(Y, \sigma)$ are topological spaces. A function is said to be g*s-closed function if for each closed set $F$ in $(X, T)$ then $f(F)$ g*s-closed in $(Y, \sigma)$.

Now, the following remark shows the relation between g*s-closed function and closed function.

Remark (2.2), [1]:-
Every closed function is g*s-closed function, but the converse is not necessary to be true. To illustrate that consider the following example.

Let $X = Y = \{a, b, c\}$ and defined the topologies $(X, T) = \{X, \phi, \{b, c\}\}$ and $(Y, \sigma) = \{X, \phi, \{a\}\}$. The function $f : (X, T) \longrightarrow (Y, \sigma)$ defined by $f(x) = x$ is g*s-closed function but not closed since $\{a\}$ is closed in $(X, T)$ and $f(\{a\}) = \{a\}$ is not closed in $(Y, \sigma)$. 

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Now, we give other type of $g^s$-closed functions from modification of definition (2.1) which call strongly $g^s$-closed function.

**Definition (2.3):**
Let $(X, T)$ and $(Y, \sigma)$ are topological spaces. A function $f : (X, T) \rightarrow (Y, \sigma)$ is said to be strongly $g^s$-closed function if for each $g^s$-closed set $F$ in $(X, T)$ then $f(F)$ is closed in $(Y, \sigma)$.

Next, the following proposition give the relation between strongly $g^s$-closed function and closed function.

**Proposition (2.4):**
Let $f : (X, T) \rightarrow (Y, \sigma)$ be a function. If $f$ is strongly $g^s$-closed function then $f$ is closed function.

**Proof:**
Let $f$ is strongly $g^s$-closed function and $F$ is closed set in $(X, T)$ by using remark (1.4), we get $F$ is $g^s$-closed set, thus $f(F)$ is closed set in $(Y, \sigma)$, then $f$ is closed function.

But, the converse of proposition above is not necessary to be true. To illustrate that consider the following example.

**Example (2.5):**
Let $X = Y = \{a, b\}$, $(X, T) = \{X, \phi\}$ and $(Y, \sigma) = \{X, \phi, \{b\}\}$, then the function $f : (X, T) \rightarrow (Y, \sigma)$ is defined by $f(x) = x$ is closed function but not strongly $g^s$-closed function since $\{b\}$ is $g^s$-closed in $(X, T)$ but not closed in $(Y, \sigma)$.

From proposition (2.4), we can get the following corollary. The proof is easy thus we its omitted

**Corollary (2.6):**
Let $f : (X, T) \rightarrow (Y, \sigma)$ be a function. If $f$ is strongly $g^s$-closed function then $f$ is $g^s$-closed function.

also, the converse of above corollary is not necessary to be true. To show that see the following example.

**Example (2.7):**
Let $X = Y = \{a, b\}$, $(X, T)$ an indiscrete topology on $X$ and $(Y, \sigma) = \{X, \phi, \{a\}\}$, then the function $f : (X, T) \rightarrow (Y, \sigma)$ the identity function is $g^s$-closed function but not strongly $g^s$-closed function since $\{a\}$ is $g^s$-closed in $(X, T)$ but not closed in $(Y, \sigma)$. 

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More, properties of strongly g*-closed function have been given by the following theorem

**Theorem (2.8):**

Let \( f : (X, T) \rightarrow (Y, \sigma) \) be a surjective function. Then \( f \) is strongly g*-s-closed function if and only if for each subset \( S \) of \( Y \) and each g*-s-open \( U \) in \( (X, T) \) there is an open set \( V \) in \( (Y, \sigma) \) such that \( S \subseteq V \) and \( f^{-1}(V) \subseteq U \).

**Proof:**

**Necessity.** Suppose that \( f \) is strongly g*-s-closed function and let \( S \) subset of \( Y \) and each g*-s-open \( U \) in \( X \) containing \( f^{-1}(S) \). PUT \( V = Y - (f(X - U)) \), then \( S \subseteq V \) and \( f^{-1}(V) \subseteq U \).

**Sufficiency.** Let \( F \) be g*-s-closed set in \( X \), then \( f^{-1}(Y - f(F)) \subseteq X - F \) and \( X - F \) is g*-open then by hypothesis there is open set \( V \) in \( (Y, \sigma) \) such that \( Y - f(F) \subseteq V \) and \( f^{-1}(V) \subseteq X - F \). Therefore, we have \( Y - V \subseteq f(F) \) and \( F \subseteq f^{-1}(Y - V) \), hence we obtain \( f(F) = Y - V \), thus \( f(F) \) is closed set in \( (Y, \sigma) \) then \( f \) is strongly g*-s-closed function.

As well as these types can be modified g*-s-closed function into the following definition

**Definition (2.9):**

Let \((X, T)\) and \((Y, \sigma)\) are topological spaces. A function \( f : (X, T) \rightarrow (Y, \sigma) \) is said to be irresolute g*-s-closed function if for each g*-s-closed set \( F \) in \( (X, T) \) then \( f(F) \) is g*-s- closed in \( (Y, \sigma) \).

Next, the following proposition shows the relationship of irresolute g*-s-closed with other types of these functions

**Proposition (2.10):**

Let \( f : (X, T) \rightarrow (Y, \sigma) \) be a function. If \( f \) is strongly g*-s-closed function then \( f \) is irresolute g*-s- closed function.

**Proof:**

Let \( f \) is strongly g*-s-closed function and \( F \) is g*-s- closed set in \( (X, T) \) the using definition (2.3), we get \( f(F) \) is closed set, thus \( f(F) \) is g*-s-closed set in \( (Y, \sigma) \), then \( f \) is irresolute g*-s- closed function.

But, the converse of above proposition above is not necessary to be true. To illustrate that consider the following example.

**Example (2.11):**

Let \( X = Y = \{a, b\} \), \( (X, T) = \{X, \phi\} \) and \( (Y, \sigma) = \{X, \phi, \{b\}, \{a\}\} \), then the function \( f : (X, T) \rightarrow (Y, \sigma) \) is defined by \( f(x) = x \) is closed function but not
irresolute $g^s$-closed function since $\{b\}$ is $g^s$-closed in $(X,T)$ but not $g^s$-closed in $(Y,\sigma)$.

also, give the relation between irresolute $g^s$-closed function and $g^s$-closed function by the following proposition.

**Proposition (2.12):**

Let $f : (X,T) \rightarrow (Y,\sigma)$ be a function. If $f$ is irresolute $g^s$-closed function then $f$ is $g^s$-closed function.

**Proof:**

Let $f$ is irresolute $g^s$-closed function and $F$ is closed set in $(X,T)$ the using remark (1.4), we get $f(F)$ is $g^s$- closed set, thus $f(F)$ is $g^s$-closed set in $(Y,\sigma)$, then $f$ is $g^s$-closed function.

also, the converse of above proposition above is not necessary to be true. To illustrate that consider the following example.

**Example (2.11):**

Let $X = Y = \{a,b\}$, $(X,T) = \{X,\emptyset,\{a\}\}$ and $(Y,\sigma) = \{Y,\emptyset,\{b\}\}$, then the function $f : (X,T) \rightarrow (Y,\sigma)$ is defined by $f(x) = x$ is closed function but not $g^s$-closed function since $\{b\}$ is closed in $(X,T)$ but not $g^s$-closed set in $(Y,\sigma)$.

The following theorem give another properties of irresolute $g^s$-closed function.

**Theorem (2.12):**

Let $f : (X,T) \rightarrow (Y,\sigma)$ be a surjective function. Then $f$ is irresolute $g^s$-closed function if and only if for each subset $S$ of $Y$ and each $g^s$-open $U$ in $(X,T)$ there is an $g^s$-open set $V$ in $(Y,\sigma)$ such that $S \subseteq V$ and $f^{-1}(V) \subseteq U$.

**Proof:**

Necessity. Suppose that $f$ is irresolute $g^s$-closed function and let subset $S$ of $Y$ and each $g^s$-open $U$ in $X$ containing $f^{-1}(S)$. PUT $V = Y - (f(X - U))$, then $S \subseteq V$ and $f^{-1}(V) \subseteq U$.

Sufficiency. Let $F$ be $g^s$-closed set in $X$, then $f^{-1}(Y - f(F)) \subseteq X - F$ and $X - F$ is $g^s$-open then by hypothesis there is open set $V$ in $(Y,\sigma)$ such that $Y - f(F) \subseteq V$ and $f^{-1}(V) \subseteq X - F$. Therefore, we have $Y - V \subseteq f(F)$ and $F \subseteq f^{-1}(Y - V)$, hence we obtain $f(F) = Y - V$, thus $f(F)$ is $g^s$-closed set in $(Y,\sigma)$ then $f$ is irresolute $g^s$-closed function.

**The Composition of Some Types Of $g^s$-Closed Functions:**

In this section, we introduce the composition of some types of $g^s$-closed function which studied in previous section.
Theorem (3.1):-
Let \((X,T),(Y,\sigma)\) and \((Z,\eta)\) topological spaces and \(f : (X,T)\longrightarrow(Y,\sigma)\) be \(g^*\)-closed function.

1. if \(h : (Y,\sigma)\longrightarrow(Z,\eta)\) is strongly \(g^*\)-closed then \(hof\) is closed function.
2. if \(h : (Y,\sigma)\longrightarrow(Z,\eta)\) is irresolute \(g^*\)-closed then \(hof\) is \(g^*\)-closed function.

Proof:-
1. Let \(F\) be a closed set in \((X,T)\) and since \(f\) is \(g^*\)-closed function then \(f(F)\) is \(g^*\)-closed set in \(Y\) also, since \(h\) is strongly \(g^*\) – closed function thus, \(h(f(F))\) is closed set in \(Z\), but \(h(f(F)) = (hof)(F)\) Therefore; \(hof\) is closed function.
2. Let \(F\) be a closed set in \((X,T)\) and since \(f\) is \(g^*\)-closed function then \(f(F)\) is \(g^*\)-closed set in \(Y\) also, since \(h\) is irresolute \(g^*\) – closed function thus, \(h(f(F))\) is \(g^*\)-closed set in \(Z\), but \(h(f(F)) = (hof)(F)\) Therefore; \(hof\) is \(g^*\)-closed function.

Theorem (3.2):-
Let \((X,T),(Y,\sigma)\) and \((Z,\eta)\) topological spaces and \(f : (X,T)\longrightarrow(Y,\sigma)\) be closed function.

1. if \(h : (Y,\sigma)\longrightarrow(Z,\eta)\) is \(g^*\)-closed function then \(hof\) is \(g^*\)-closed function.
2. if \(h : (Y,\sigma)\longrightarrow(Z,\eta)\) is strongly \(g^*\)-closed function then \(hof\) is closed function.
3. if \(h : (Y,\sigma)\longrightarrow(Z,\eta)\) is irresolute \(g^*\)-closed function \(hof\) is \(g^*\)-closed function.

Proof:-
1. Let \(F\) be a closed set in \((X,T)\) and since \(f\) is closed function then \(f(F)\) is closed set in \(Y\) also, since \(h\) is \(g^*\) – closed function thus, \(h(f(F))\) is \(g^*\)-closed set in \(Z\), but \(h(f(F)) = (hof)(F)\) Therefore; \(hof\) is \(g^*\)-closed function.
2. Let \(F\) be a closed set in \((X,T)\) and since \(f\) is closed function then \(f(F)\) is closed set in \(Y\) and by remark (1.4) we get \(f(F)\) is \(g^*\)-closed set also, since \(h\) is strongly \(g^*\) – closed function thus, \(h(f(F))\) is closed set in \(Z\), but \(h(f(F)) = (hof)(F)\) Therefore; \(hof\) is \(g^*\)-closed function.
3. Let \(F\) be a closed set in \((X,T)\) and since \(f\) is closed function then \(f(F)\) is closed set in \(Y\) and by remark (1.4) we get \(f(F)\) is \(g^*\)-closed set also, since \(h\) is irresolute \(g^*\) – closed function thus, \(h(f(F))\) is \(g^*\)-closed set in \(Z\), but \(h(f(F)) = (hof)(F)\) Therefore; \(hof\) is \(g^*\)-closed function.
Theorem (3.3):-
Let \((X, T), (Y, \sigma), (Z, \eta)\) topological spaces and \(f : (X, T) \longrightarrow (Y, \sigma)\) be strongly g*-closed function.

1. If \(h : (Y, \sigma) \longrightarrow (Z, \eta)\) is g*-closed function then \(hof\) is irresolute g*-closed function.
2. If \(h : (Y, \sigma) \longrightarrow (Z, \eta)\) is closed function then \(hof\) is strongly g*-closed function.
3. If \(h : (Y, \sigma) \longrightarrow (Z, \eta)\) is strongly g*-closed function then \(hof\) is strongly g*-closed function.
4. If \(h : (Y, \sigma) \longrightarrow (Z, \eta)\) is irresolute g*-closed function then \(hof\) is irresolute g*-closed function.

Proof:-
1. Let \(F\) be a g*-closed set in \((X, T)\) and since \(f\) is strongly g*-closed function then \(f(F)\) is closed set in \(Y\) also, since \(h\) is g*-closed function thus, \(h(f(F))\) is g*-closed set in \(Z\), but \(h(f(F)) = (hof)(F)\) Therefore; \(hof\) is irresolute g*-closed function.
2. Let \(F\) be a g*-closed set in \((X, T)\) and since \(f\) is strongly g*-closed function then \(f(F)\) is closed set in \(Y\) also, since \(h\) is closed function thus, \(h(f(F))\) is closed set in \(Z\), but \(h(f(F)) = (hof)(F)\) Therefore; \(hof\) is strongly g*-closed function.
3. Let \(F\) be a g*-closed set in \((X, T)\) and since \(f\) is strongly g*-closed function then \(f(F)\) is closed set in \(Y\) and by remark (1.4) we get \(f(F)\) is g*-closed set also, since \(h\) is strongly g*-closed function thus, \(h(f(F))\) is closed set in \(Z\), but \(h(f(F)) = (hof)(F)\) Therefore; \(hof\) is irresolute g*-closed function.
4. Let \(F\) be a g*-closed set in \((X, T)\) and since \(f\) is strongly g*-closed function then \(f(F)\) is closed set in \(Y\) and by remark (1.4) we get \(f(F)\) is g*-closed set also, since \(h\) is irresolute g*-closed function thus, \(h(f(F))\) is g*-closed set in \(Z\), but \(h(f(F)) = (hof)(F)\) Therefore; \(hof\) is irresolute g*-closed function.

Theorem (3.4):-
Let \((X, T), (Y, \sigma), (Z, \eta)\) topological spaces and \(f : (X, T) \longrightarrow (Y, \sigma)\) be irresolute g*-closed function.

1. If \(h : (Y, \sigma) \longrightarrow (Z, \eta)\) is strongly g*-closed function then \(hof\) is strongly g*-closed function.
2. If \(h : (Y, \sigma) \longrightarrow (Z, \eta)\) is irresolute g*-closed function then \(hof\) is irresolute g*-closed function.

Proof:-
1. Let \(F\) be a g*-closed set in \((X, T)\) and since \(f\) is irresolute g*-closed function then \(f(F)\) is g*-closed set in \(Y\) also, since \(h\) is strongly g*-closed function thus,
\( h(f(F)) \) is closed set in \( Z \), but \( h(f(F)) = (hof)(F) \) Therefore; \( hof \) is strongly \( g^*s \) – closed function.

(2) Let \( F \) be a \( g^*s \)- closed set in \( (X,T) \) and since \( f \) is irresolute \( g^*s \)-closed function then \( f(F) \) is \( g^*s \)-closed set in \( Y \) also, since \( h \) is irresolute \( g^*s \) – closed function thus, \( h(f(F)) \) is closed set in \( Z \), but \( h(f(F)) = (hof)(F) \) Therefore; \( hof \) is irresolute \( g^*s \) – closed function.

References: