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# Continuous and Uniform Continuous Mappings on a Standard Fuzzy Metric Spaces

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## ABSTRACT

In this paper we introduced the definition of standard fuzzy metric spaces then we discussed several properties of this space after some illustrative examples are given. Then we defined a continuous mapping from standard fuzzy metric space $(X,M_X,*)$ into a standard fuzzy metric space $(Y,M_Y,*)$ after that we proved some basic theorems of a continuous mappings. Finally we defined uniformly continuous mapping from a standard fuzzy metric space $(X,M_X,*)$ into a standard fuzzy metric space $(Y,M_Y,*)$ then we proved several properties of uniformly continuous mapping.

**Keyword:** Standard fuzzy metric space**,** Continuous mapping, Uniformly continuous mapping.

## الدوال المستمرة والدوال المنتظمة الاستمرارية على الفضاءات المترية الضبابية القياسية

**الخلاصة**

في هذا البحث قدمنا تعريف الفضاء المتري الضبابي القياسي بعد ذلك ناقشنا عدة خواص لهذا الفضاء بعد أن أعطينا امثلة توضيحية . بعد ذلك عرفنا الدوال المستمرة من فضاء متري ضبابي قياسي $(X,M_X,*)$ الى فضاء متري ضبابي قياسي أخر $(Y,M_Y,*)$ ثم برهنا بعض المبرهنات الاساسية للدوال المستمرة واخيرا" عرفنا الدوال المستمرة المنتظمة من فضاء متري ضبابي قياسي $(X,M_X,*)$ الى فضاء متري ضبابي قياسي أخر $(Y,M_Y,*)$ بعد ذلك برهنا عدة خواص للدوال المستمرة المنتظمة .

## INTRODUCTION

Many authors have introduced and discussed several notions of fuzzy metric space from different points of view [1, 2, 8,10,11 and 12]. In particular, Kramosil and Michalek [7] generalized the concept of probabilistic metric space given by Menger [8], [11] to the fuzzy framework. Later on, George and Veeramani [2] have modified in a slight but appealing way the concept of fuzzy metric space of Kramosil and Michalek. Other recent contributions to the study of fuzzy metric spaces in the sense of [2] may be found in [3,4,5,6 and 9]. Here we introduce a new definition of fuzzy metric space we call it a standard fuzzy metric space then we give two examples to illustrated this notion after that in section one we explore several properties of this space. One of



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the main aims in considering standard fuzzy metric spaces is the study of continuous functions. In section two we define continuous mapping from a standard fuzzy metric space $(X,M_X,*)$ into a standard fuzzy metric space $(Y,M_Y,*)$ then we proved several properties of a continuous mapping. In section three we define a uniformly continuous mapping from a standard fuzzy metric space $(X,M_X,*)$ into a standard fuzzy metric space $(Y,M_Y,*)$.
Every function which is uniformly continuous is necessarily continuous we give a counter example to show that converse may not be true. Also we prove that uniformly continuous mapping maps a Cauchy sequence into a Cauchy sequence and we give an example to show that continuous mapping does not maps Cauchy sequence to a Cauchy sequence.

## Standard fuzzy metric spaces

**Definition 1.1:[1] A binary operation** $*: [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions:

1- $*$ is associative and commutative.
2- $*$ is continuous.
3- $a*1 = a$ for all $a \in [0,1]$.
4- $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$ where $a,b, c,d \in[0,1]$.

**Remark 1.2:[2]** For any $r_1 > r_2$ we can find $r_3$ such that $r_1* r_2 \geq r_2$ and for any $r_4$ we can find an $r_5$ such that $r_5*r_5 \geq r_4$ where $r_1,r_2,r_3,r_4,r_5 \in(0,1)$.
We introduce the following definition.
**Definition 1.3:** A triple $(X,M,*)$ is said to be standard fuzzy metric space if X is an arbitrary set, $*$ is a continuous t- norm and M is a fuzzy set on $X^2$ satisfying the following conditions:
$(FM_1)$ $M(x,y) > 0$ for all $x, y \in X$
$(FM_2)$ $M(x,y) = 1$ if and only if $x = y$
$(FM_3)$ $M(x,y) = M(y,x)$ for all $x, y \in X$
$(FM_4)$ $M(x,z) \geq M(x,y) *M(y,z)$ for all x, y and $z \in X$
$(FM_5)$ $M(x,y)$ is a continuous fuzzy set

**Example 1.4:** Let $X= \mathbb{N}$, and let $a*b = a.b$ for all $a, b \in[0,1]$.

$$M(x,y) = \begin{cases} \frac{x}{y} & \text{If } x \leq y \\ \frac{y}{x} & \text{If } y \leq x \end{cases}$$

for all $x, y \in\mathbb{N}$.
Then it is easy to show that $(\mathbb{N},M,.)$ is a standard fuzzy metric space.
**Example 1.5:** Let $X=\mathbb{R}$ and let $a*b = a.b$ for all $a, b \in [0,1]$.
Define $M(x,y) = \frac{1}{e^{|x-y|}}$ for all $x, y \in \mathbb{R}$ .
Then it is easy to show that $(\mathbb{R},M,.)$ is a standard fuzzy metric space.

**Definition 1.6:** Let $(X,M,*)$ be a standard fuzzy metric space then M is continuous if whenever $x_n \to x$ and $y_n \to y$ in X then $M(x_n,y_n) \to M(x,y)$ that is $\lim_{n\to\infty} M(x_n,y_n) = M(x,y)$.

**Definition 1.7:** Let $(X,M,*)$ be a standard fuzzy metric space .Then $B(x,r) = \{y\in X: M(x,y) > 1-r\}$ is an open ball with center $x\in X$ and radius r, $0 < r < 1$**.**

The proof of the following result is easy and hence is omitted**.**

**Proposition 1.8:** Let $B(x,r_1)$ and $B(x,r_2)$ be two open balls with same center x in a standard fuzzy metric space $(X,M,*)$.Then either

$B(x,r_1) \subseteq B(x,r_2)$ or $B(x,r_2)\subseteq B(x,r_1)$ where $r_1,r_2 \in (0,1)$.

**Definition 1.9:** A subset A of a standard fuzzy metric space $(X,M,*)$ is said to be open if given any point a in A there exists r , $0 < r < 1$ such that $B(a,r) \subseteq A$ .A subset B is said to be closed if $B^c$ is open .

The idea of the proof of the following result is similar to the idea of result 3.2 in [2] and hence is omitted.

**Theorem 1.10:** Every open ball in a standard fuzzy metric space $(X,M,*)$ is an open set.

The proof of the following result is easy, hence is omitted.

**Theorem 1.11:** Let $(X,M,*)$ is a standard fuzzy metric space. Define $\tau_M = \{A\subset X$: $x\in A$ if and only if there exists $0 < r < 1$ such that $B(x,r) \subset A\}$ Then $\tau_M$ is a topology on X.

**Theorem 1.12:** Every standard fuzzy metric space is a Hausdorff space.

**Proof :**

Let $(X,M,*)$ be a standard fuzzy metric space. Let x and y be two distinct pints of X. Then $0 < M(x,y) < 1$ .Let $M(x,y) = r$ , for some $0<r<1$. Now by Remark 2.2 for each $r_0$, $r < r_0 < 1$, we can find an $r_1$ such that $r_1 * r_1 \geq r_0$ . Consider the open balls $B(x,1-r_1)$ and $B(y,1-r_1)$.

Clearly $B(x,1-r_1) \cap B(y,1-r_1) = \emptyset$ for if there exists

$z\in B(x,1-r_1) \cap B(y,1-r_1)$.

Then $r = M(x,y) \geq M(x,z) * M(z,y) \geq r_1 * r_1 \geq r_0 > r$ which is a contradiction. Therefore $(X,M,*)$ is a Hausdorff space.

**Definition 1.13:** A sequence $(x_n)$ in a standard fuzzy metric space $(X,M,*)$ is said to be converge to a point x in X if for each r ,$0 < r < 1$ there exists a positive number N such that $M(x_n,x) > (1-r)$ , for each $n \geq N$ .

The idea of the proof of the following theorem is similar to the idea of the proof of Theorem 3.11 in [2] and hence is omitted.

**Theorem 1.14:** Let $(X,M,*)$ be a standard fuzzy metric space then for a sequence $(x_n)$ in X converge to x if and only if $\lim_{n\to\infty} M(x_n,x) = 1$.

**Definition 1.15:** A sequence $(x_n)$ in a standard fuzzy metric space $(X,M,*)$ is Cauchy if for each r, $0 < r < 1$, there exists a positive number N such that $M(x_n,x_m) > (1-r)$, for each $m, n \geq N$.

**Lemma1.16:[13]** Let $f: X\to Y$ be an arbitrary function and $A \subseteq X$ and $B \subseteq Y$. Then $f(A) \subseteq B$ if and only if $A \subseteq f^{-1}(B)$.

**Definition1.17:[13]** Let X and Y be a standard fuzzy metric spaces and let A be a proper subset of X. If f is a mapping of A into Y, then a mapping $g:X\to Y$ is called an extension of f if $g(x) = f(x)$ for each $x\in A$, the function f is then called the restriction of g to A.

**Definition1.18:** A standard fuzzy metric space $(X,M,*)$ is complete if every Cauchy sequence in X converges to a point in X.

**Continuous mappings**

**Definition 2.1:** Let $(X,M_X,*)$ and $(Y,M_Y,*)$ be standard fuzzy metric spaces and $A \subseteq X$. A function $f:A\to Y$ is said to be continuous at $a\in A$, if for every $0 < \varepsilon < 1$, there exist some $0 < \delta < 1$, such that $M_Y(f(x),f(a)) > (1-\varepsilon)$ whenever $x\in A$ and $M_X(x,a) > (1-\delta)$. If f is continuous at every point of A, then it is said to be continuous on A.

**Theorem 2.2:** Let $(X,M_X,*)$ and $(Y,M_Y,*)$ be standard fuzzy metric spaces and $A\subseteq X$. A function $f:A\to Y$ is continuous at $a\in A$ if and only if whenever a sequence $(x_n)$ in A converge to a, the sequence $(f(x_n))$ converges to f(a).

**Proof:** First suppose the function $f:A\to Y$ is continuous at $a\in A$ and let $(x_n)$ be a sequence in A converge to a. We shall show that $(f(x_n))$ converges to f(a). Let $0 < \varepsilon < 1$ be given. By continuity of f at a, there exists some $0 < \delta < 1$, such that $x\in A$ and $M_X(x,a) > (1-\delta)$, implies
$M_Y(f(x),f(a)) > (1-\varepsilon)$. Since $\lim_{n\to\infty} x_n = a$ there exist some positive integer N such that $n \geq N$ implies $M_X(x_n,a) > (1-\delta)$. Therefore $n \geq N$ implies $M_Y(f(x_n),f(a)) > (1-\varepsilon)$. Thus $\lim_{n\to\infty} f(x_n) = f(a)$

Now suppose that every sequence $(x_n)$ in A converging to a has the property that $\lim_{n\to\infty} f(x_n) = f(a)$. We shall show that f is continuous at a.

Suppose, if possible, that f is not continuous at a. There must exists
$0 < \varepsilon < 1$, for which no $\delta$, $0 < \delta < 1$ can satisfy the requirement that $x\in A$ and $M_X(x,a) > (1-\delta)$, implies $M_Y(f(x),f(a)) > (1-\varepsilon)$. This means that for every $\delta$, $0 < \delta < 1$ there exists $x\in A$ such that $M_X(x,a) > (1-\delta)$ but $M_Y(f(x),f(a)) \leq (1-\varepsilon)$. For every $n\in N$, the number $\frac{1}{n}$ is positive and therefore there exist $x_n\in A$ such that $M_X(x_n,a) > (1-\frac{1}{n})$ but
$M_Y(f(x_n),f(a)) \leq (1-\varepsilon)$. The sequence $(x_n)$ then converges to a but the sequence $(f(x_n))$ dose not converge to f(a). This contradicts the assumption that every sequence $(x_n)$ in A converging to a has the property that $\lim_{n\to\infty} f(x_n) = f(a)$. Therefore, the supposition that f is not continuous at a must be false■

**Definition 2.3:** Let $(X,M_X,*)$ and $(Y,M_Y,*)$ be standard fuzzy metric spaces and $A \subseteq X$. Let $f:A\to X$ and a be a limit point of A. We write $\lim_{x\to a} f(x) = b$, where $b\in Y$, if for every $0 < \varepsilon < 1$ there exists $0 < \delta < 1$ such that $M_Y(f(x),b) > 1-\varepsilon$ wherever $x\in A$ and $M_X(x,a) > 1-\delta$.

**Proposition 2.4:** Let $(X,M_X,*)$ and $(Y,M_Y,*)$ , A, f and a be as in Definition 2.3. Then $\lim_{x\to a} f(x) = b$ if and only if $\lim_{n\to\infty} f(x_n) = b$ for every sequence $(x_n)$ in A such that $x_n \neq a$ and $\lim_{n\to\infty} x_n = a$.

**Proof:**

The argument is similar to that of Theorem 2.2 and is therefore not included.

**Proposition 2.5:** A mapping f of a standard fuzzy metric space $(X,M_X,*)$ into a standard fuzzy metric space $(Y,M_Y,*)$ is continuous at a point $a\in X$ if and only if for every $0 < \varepsilon < 1$, there exists $0 < \delta < 1$ such that , $B(a,\delta) \subseteq f^{-1}(B(f(a),\varepsilon))$ where B(x,r) denotes the open ball of radius r with center x.

**Proof:**

The mapping $f:X\to Y$ is continuous at $a\in X$ if and only if for every

$0 < \varepsilon < 1$, there exists $0 < \delta < 1$ such that $M_Y(f(x),f(a)) > (1-\varepsilon)$ for all x satisfying $M_X(x,a) > 1-\delta$ i.e $x \in B(a,\delta)$ implies $f(x) \in B(f(a),\varepsilon)$ or

$$f(B(a,\delta)) \subseteq B(f(a),\varepsilon)$$

This is equivalent to the condition

$$B(a,\delta) \subseteq f^{-1}(B(f(a),\varepsilon)) \blacksquare$$

**Theorem 2.6:** A mapping $f{:}X \to Y$ is continuous on X if and only if $f^{-1}(G)$ is open in X for all open subset G of Y.

**Proof:**

Suppose f is continuous on X and let G be an open subset of Y.

We have to show $f^{-1}(G)$ is open in X. Since $\emptyset$ and X are open, we may suppose that $f^{-1}(G) \neq \emptyset$ and $f^{-1}(G) \neq X$. Let $x \in f^{-1}(G)$. Then $f(x) \in G$. Since G is open, there exists $0 < \varepsilon < 1$ such that $B(f(x),\varepsilon) \subseteq G$. Since f is continuous at x, by Proposition 1.4.6 for this ε there exists $0 < \delta < 1$ such that $B(x,\delta) \subseteq f^{-1}(B(f(x),\varepsilon)) \subseteq f^{-1}(G)$

Thus, every point x of $f^{-1}(G)$ is an interior point, and so $f^{-1}(G)$ is open in X. Suppose, conversely, that $f^{-1}(G)$ is open in X for all open subsets G of Y. Let $x \in X$ for each $0 < \varepsilon < 1$, the set $B(f(x),\varepsilon)$ is open and so $f^{-1}(B(f(x),\varepsilon))$ is open in X. Since $x \in f^{-1}(B(f(x),\varepsilon))$ it follows that there exists $0 < \delta < 1$ such that $B(x,\delta) \subseteq f^{-1}(B(f(x),\varepsilon))$.

By Proposition 2.5 it follows that f is continuous of x $\blacksquare$

**Theorem 2.7:**

A mapping $f{:}X \to Y$ is continuous on X if and only if $f^{-1}(F)$ is closed in X for all closed subset F of Y.

**Proof:**

Let F be a closed subset of Y. Then Y-F is open in Y so that $f^{-1}(Y-F)$ is open in X by Theorem 2.6. But $f^{-1}(Y-F) = X - f^{-1}(F)$ so $f^{-1}(F)$ is closed in X. Suppose, conversely, that $f^{-1}(F)$ is closed in X for all closed subset F of Y. But the empty set and the whole space X are closed sets. Then $X - f^{-1}(F)$ is open in X and $f^{-1}(Y-F) = X - f^{-1}(F)$ is open in X. Since every open subset of Y is of the type Y- F where F is suitable closed set. It follows by using Theorem 2.6, that f is continuous$\blacksquare$

The characterization of continuity in terms of open sets of Theorem 2.6 leads to an elegant and brief proof of the fact that a composition of continuous maps is continuous.The idea of the proof of the following theorem is similar to the idea of the proof of the ordinary case .

**Theorem 2.8:** Let $(X,M_X,*)$, $(Y,M_Y,*)$ and $(Z,M_Z,*)$ be a standard fuzzy metric spaces and let $f{:}X \to Y$ and $g{:}Y \to Z$ be continuous. Then the composition $g \circ f$ is a continuous map of X into Z.

**Proof:**

Let G be open subset of Z. By Theorem 2.6, $g^{-1}(G)$ is an open subset of Y and another application of the same theorem shows that $f^{-1}(g^{-1}(G))$ is an open subset of X. Since $(g \circ f)^{-1}(G) = f^{-1}(g^{-1}(G))$, it follows from the same theorem again that $g \circ f$ is continuous $\blacksquare$

The idea of the proof of the following theorem is similar to the idea of the proof of the ordinary case .

**Theorem 2.9:** Let $(X, M_X,*)$ and $(Y,M_Y,*)$ be a standard fuzzy metric spaces and let $f{:}X \to Y$. Then the following statements are equivalent:

(i) f is continuous on X.
(ii) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for all subsets B of Y.
(iii) $f(\overline{A}) \subseteq \overline{f(A)}$ for all subsets A of X.

**Proof:**

**(i)⇒(ii):** Let B be a subset of Y. Since $\overline{B}$ is a closed subset of Y, $f^{-1}(\overline{B})$ is closed in X. Moreover $f^{-1}(B) \subseteq f^{-1}(\overline{B})$, and so $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$. [Recall that $\overline{f^{-1}(B)}$ is the smallest closed set containing $f^{-1}(B)$].

**(ii)⇒(iii):** Let A be a subset of X. Then if B = f(A), we have $A \subseteq f^{-1}(B)$ and $\overline{A} \subseteq \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$. Thus $f(\overline{A}) \subseteq f(f^{-1}(\overline{B})) = \overline{B} = \overline{f(A)}$.

**(iii)⇒(i):** Let F be a closed set in Y and set $f^{-1}(F) = F_1$. By Theorem 2.7, it is sufficient to show that $F_1$ is closed in X, that is, $F_1 = \overline{F_1}$.

Now $f(\overline{F_1}) \subseteq \overline{f(f^{-1}(F))} \subseteq \overline{F} = F$ so that $\overline{F_1} \subseteq f^{-1}(f(\overline{F_1})) \subseteq f^{-1}(F) = F_1$ ■

**Definition 2.10:** Let X and Y be a standard fuzzy metric spaces and let A be a proper subset of X. If f is a mapping of A into Y, then a mapping g:X→Y is called an extension of f if g(x) = f(x) for each x∈A, the function f is then called the restriction of g to A.

If X and Y are standard fuzzy metric spaces, A ⊂ X and f:A→Y is continuous, then we might ask whether there exists a continuous extension g of f. Extension problems abound in analysis and have attracted the attention of many celebrated mathematicians. Below we deal with some simple extension techniques.

**Theorem 2.11:** Let $(X,M_X,*)$ and $(Y,M_Y,*)$ be two standard fuzzy metric spaces. Let f:X→Y and g:X→Y be continuous mappings. Then the set $\{x\in X: M_Y(f(x),g(x)) = 1\}$ is closed subset of X.

**Proof:**

Let $F = \{x\in X: M_Y(f(x),g(x)) = 1\}$. Then $X-F = \{x\in X: 0 < M_Y(f(x),g(x)) < 1\}$. We show that X-F is open. If X- F = ∅, then there is nothing to prove. So let X-F ≠ ∅ and let a∈X-F . Then $M_Y(f(a),g(a)) < 1$.

Let $M_Y(f(a),g(a)) = 1-\varepsilon$, for some $0 < \varepsilon < 1$ . Then by continuity of f and g there is $0 < \delta < 1$ such that $M_x(x,a) > 1-\delta$ . Implies $M_Y(f(x),f(a)) > 1-\varepsilon$ and $M_Y(g(x),g(a)) > 1-\varepsilon$. Hence there exists (1- r) for some $0 < r < 1$ . By Remark 1.2, such that

$$(1-\varepsilon)*(1-\varepsilon)*(1-\varepsilon) > (1-r)$$

Now $M_Y(f(x),g(x)) \geq M(f(x),f(a))*M(f(a),g(a))*M(g(a),g(x))$

$$> (1-\varepsilon)*(1-\varepsilon)*(1-\varepsilon) > (1-r)$$

For all x satisfying $M_x(x,a) > (1-\delta)$.

Thus for each $x\in B(a,\delta)$, $M_Y(f(x),g(x)) < 1$ , i.e, f(x) ≠ g(x).

So $B(a,\delta) \subseteq X-F$ . Hence, X-F is open and thus F is closed ■

**Corollary 2.12:** Let $(X,M_X,*)$ and $(Y,M_Y,*)$ be standard fuzzy metric spaces. Let f:X→Y and g:X→Y be continuous mappings. If $F = \{x\in X: M_Y(f(x),g(x)) =1\}$ is dense in X then f = g.

**Proof:**

By Theorem 2.11, F is closed. Since F is assumed dense in X, we have $X = \overline{F} = F$ i.e f(x) = g(x) for all x∈X■

**Theorem 2.13:** Let $(X,M_X,*)$ and $(Y,M_Y,*)$ be standard fuzzy metric spaces. Let A be a dense subset of X and f:A→Y be a map. Then f has a continuous extension g:X→Y if and only if for every x∈X that is a limit point of A, the limit $\lim_{y\to x} f(y)$ not only exists in Y but also equals f(x) in case x∈A. When the extension exists, it is unique. [Note that the stipulation $\lim_{y\to x} f(y) = f(x)$ when x∈A says that f is continuous on A].

**Proof:**

Suppose that f has a continuous extension g, and consider any $x \in X$ that is a limit point of X. Since A is dense, x must be a limit point of A as well, as we now argue. Any ball $B(x,\varepsilon)$ contains a point $y \in X$, $y \neq x$.

There exists $B(y,\acute{\varepsilon}) \subseteq B(x,\varepsilon)$ such that $x \notin B(y,\acute{\varepsilon})$. Since A is dense, $B(y,\acute{\varepsilon})$ contains a point $a \in A$. Thus $B(x,\varepsilon)$ contains the point $a \in A$ and $a \neq x$. Now

$$\begin{aligned} g(x) &= \lim_{y\to x} g(y) && [\text{ g is continuous }] \\ &= \lim_{y\to x} g(y) && \text{with } y \in A \; [\text{ x is a limit point of A }] \\ &= \lim_{y\to x} f(y) && [\text{ g is an extension of f }] \end{aligned}$$

Thus, $\lim_{y\to x} f(y)$ exists and equals g(x).

Conversely, suppose that for every limit point $x \in X$, $\lim_{y\to x} f(y)$ exists and that is equal to f(x) when $x \in A$.

Define g(x) by:

$$g(x) = \begin{cases} f(x) & \text{if } x \in A \\ \lim_{y\to x} f(y) & \text{if } x \notin A \text{ but } x \in \acute{A} \end{cases}$$

Since A is dense in X, the function g is defined on the whole of X. We need to show that g is continuous. By the definition of a limit, for every positive number $0 < \varepsilon < 1$, there exists a positive number $0 < \delta < 1$ such that $f(y) \in B(g(x),\frac{\varepsilon}{2})$ whenever $y \neq x$ and $y \in B(x,\delta) \cap A$.Consider any $z \in B(x,\delta)$. In case z is an isolated point of X. Then $g(z) \in B(g(x),\frac{\varepsilon}{2})$ in view of the observation above. If z is not an isolated point of X, then g(z) is the limit of f(y) as $y \to z$ in $B(x,\delta) \cap A$.

Therefore $g(z) \in \overline{f(A \cap B(x,\delta))} \subseteq \overline{B(g(x),\frac{\varepsilon}{2})} \subseteq B(g(x),\varepsilon)$

So that g is continuous of x. Hence g is continuous on X. By Corollary 2.12, it follows that g is the unique continuous extension of f ■

**Uniform Continuous Mappings**

**Definition 3.1:** Let $(X,M_X,*)$ and $(Y,M_Y,*)$ be two standard fuzzy metric spaces. A function $f:X\to Y$ is said to be uniformly continuous on X, if for every $0 < \varepsilon < 1$,there exists $\delta$, $0 < \delta < 1$ (depending on $\varepsilon$ alone) such that $M_Y(f(x_1),f(x_2)) > (1-\varepsilon)$ whenever $M_x(x_1,x_2) > (1-\delta)$.

**Remark 3.2:** Every function $f:X\to Y$ which is uniformly continuous on X is necessarily continuous on X. However the converse may not be true.

**Counter Example:** Let $X = Y = \mathbb{R} - \{0\}$ and $M_x(x, y) = \frac{1}{|x-y|}$ if $x \neq y$ and $M_x(x,x) = 1$. Let $a*b = a\cdot b$ for all $a, b \in [0,1]$. Then $(X,M_X,*)$ is a standard fuzzy metric space. Let $M_Y(y_1, y_2) = \frac{1}{|y_1 - y_2|}$ and

$M_Y(y_1, y_1) = 1$, for all $y_1, y_2 \in Y$. Then $(Y,M_Y,*)$ is a standard fuzzy metric space. Define $f:X\to Y$ by $f(x) = \frac{1}{x}$, $x \in \mathbb{R} - \{0\}$. It is easy to see that f is continuous. We shall prove that it is not uniformly continuous by exhibiting an $\varepsilon$ for which no $\delta$ works. Let $0 < \delta < 1$ be arbitrary.

Choose x = $\frac{2}{1-\delta}$ and y = $\frac{1}{1-\delta}$ .

Then $M_X(x,y) = \frac{1}{|x-y|} = \frac{1}{|\frac{2}{1-\delta} - \frac{1}{1-\delta}|} = \frac{1}{\frac{1}{1-\delta}} = 1-\delta$

But $M_Y(f(x),f(y)) = \frac{1}{|f(x)-f(y)|} = \frac{1}{|\frac{1-\delta}{2} - 1 + \delta|} = \frac{2}{|1-\delta|} > 1$

Thus, whatever $0 < \delta < 1$ may be there exists points x and y such that

$M_X(x,y) > 1-\delta$ , $0 < \delta < 1$ but $M_Y(f(x),f(y)) > 1$ ■

**Remark 3.3:** A continuous function may not map a Cauchy sequence into a Cauchy sequence as the following example shows:

**Example 3.4:** Let X = $(0,\infty)$ , $a*b = a\cdot b$ for all $a,b\in[0,1]$ and let $M_X(x_1,x_2) = \frac{1}{1+|x_1-x_2|}$ , then $(X,M_X,*)$ is a standard fuzzy metric space. Let Y = $\mathbb{R}$ - {0} and let $M_Y(y_1,y_2) = \frac{1}{|y_1-y_2|}$ , if $y_1\neq y_2$ and $M_Y(y_1,y_1) = 1$. Let f: X→Y, f(x) = x, x∈X and f is a continuous function. Now $(\frac{1}{n})$ is Cauchy sequence in X because it is convergent sequence. But $(f(\frac{1}{n})) = (\frac{1}{n})$ is not a Cauchy sequence in Y. Since $M_Y(\frac{1}{m},\frac{1}{n}) = \frac{1}{|\frac{1}{m}-\frac{1}{n}|} = \frac{1}{|\frac{n-m}{mn}|} = |\frac{mn}{n-m}| > 1$ . Hence it is not Cauchy sequence in Y■

**Theorem 3.5:**Let $(X,M_X,*)$ and $(Y,M_Y,*)$be two standard fuzzy metric spaces and f:X→Y be uniformly continuous.If $(x_n)$ a Cauchy sequence in X then so is $(f(x_n))$ in Y.

**Proof:**

Since f is uniformly continuous, for every $\varepsilon > 0$, there exists δ, $0 < \delta < 1$ such that $M_Y(f(x),f(y)) > 1-\varepsilon$ whenever $M_x(x,y) > 1-\delta$ for all x, y∈X. Because the sequence $(x_n)$ is Cauchy, corresponding to $0 < \delta < 1$ there exists N such that $M_x(x_n,x_m) > 1-\delta$ for all m, n ≥ N. We now conclude that $M_Y(f(x_n),f(x_m)) > 1-\varepsilon$ for all n, m ≥ N and so $(f(x_n))$ is a Cauchy in Y□

**Theorem 3.6:** Let f be a uniformly continuous mapping of a set A, dense in the standard fuzzy metric space $(X,M_x,*)$ in to a complete standard fuzzy metric space $(Y,M_Y,*)$. Then there exists a unique continuous mapping g: X→Y such that g(x) = f(x) when x∈A. Moreover g is uniformly continuous.

**Proof:**

Since f is uniformly continuous, a fortiori, continuous, therefore, for every x∈A that is a limit point of X, the limit $\lim_{y\to x} f(y)$ not only exists in Y but also equals f(x). Therefore by Theorem 2.13, in order to prove the existence and uniqueness of such a continuous mapping g:X→Y, it is sufficient to show for every x∈X-A that f(y) tends to a limit as y→x. [ It is understood that y∈A because the domain of f is A].

Let x∈X be arbitrary. Since A is dense in X, there exists a sequence $(x_n)$ in A such that $\lim_{n\to\infty} M_X(x_n, x) = 1$. Since $(x_n)$ is convergent, it is a fortiori Cauchy so by Theorem 3.5, it follows that $(f(x_n))$ is a Cauchy sequence in the complete standard fuzzy metric space $(Y,M_Y,*)$ and hence converges to a limit, which we shall denote by b. Now consider any sequence $(x'_n)$ in A with $x'_n \neq x$ for each n and $\lim_{n\to\infty} x'_n = x$. It follows from uniform continuity of f that, for

$0 < \varepsilon < 1$, there exists a $\delta$, $0 < \delta < 1$ such that $M_Y(f(z), f(y)) > 1- \varepsilon$ whenever $M_X(z, y) > 1- \delta$ since $\lim_{n\to\infty} x_n = x = \lim_{n\to\infty} x'_n$ , there exists an integer $N_1$, such that $M_X(x_n, x'_n) > 1- \delta$ whenever $n \geq N_1$. Letting $n\to\infty$, we get
$M_Y(\lim_{n\to\infty} f(x_n), \lim_{n\to\infty} f(x'_n)) \geq 1- \varepsilon$ i.e $M_Y(b, \lim_{n\to\infty} f(x'_n)) \geq 1- \varepsilon$.
Since $0 < \varepsilon < 1$ is arbitrary, it follows that $\lim_{n\to\infty} f(x'_n) = b$, for every sequence $(x'_n)$ in A with $x'_n \neq x$ for each n, and $\lim_{n\to\infty} x'_n = x$.
It follows, Proposition 2.4, that f(y) tends to a limit, namely b, as $y\to x$.
As already pointed out earlier, this shows that a unique continuous extension g of f to X exists. It remains to prove that g is uniformly continuous. Let x and $\acute{x}$ be two points of X such that $M_X(x, \acute{x}) > (1- r)$.
Let $(x_n)$ and $(x'_n)$ be sequence of points in A such that
$\lim_{n\to\infty} M_X(x_n, x) = 1$ and $\lim_{n\to\infty} M_X(x'_n, \acute{x}) = 1$.
We can choose an integer $N_2$ such that $M_X(x_n, x) > (1- r)$
and $M_X(x'_n, \acute{x}) > (1- r)$ whenever $n \geq N_2$
Now we can find $0 < \delta < 1$ such that $(1- r) * (1- r) * (1- r) > (1- \delta)$
By Remark 1.2

$$M_X(x_m, x'_n) \geq M_X(x_m, x) * M_X(x, \acute{x}) * M_X(\acute{x}, \acute{x}_n)$$
$$\geq (1- r) * (1- r) * (1- r) > (1- \delta)$$

For all m, $n \geq N_2$, it follows that $M_Y(f(x_m), f(x'_n)) > 1- \varepsilon$
Letting $m\to\infty$ and then $n\to\infty$ in the above in equality, we get,

$$M_Y(g(x), g(\acute{x})) > 1- \varepsilon \text{ , whenever } M_X(x, \acute{x}) > (1- r)$$

This proves that g is uniformly continuous □

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