Solution of Problems in Calculus of Variations Using Parameterization Technique

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ABSTRACT

In this paper a direct method using parameterization technique is applied for solving some problems in calculus of variations. The parameterization technique based on Laguerre and Hermite polynomials is introduced to reduce a variational problem to quadratic programming problem. Examples are given to demonstrate the validity of method.

Keywords: calculus of variation, orthogonal polynomials, parameterization Technique.

حل مسائل في حسبان التغاير باستخدام تقنية تحديدالمعاييروالثوابت

الخلاصة:

في هذا البحث تم تطبيق طريقة مباشرة باستخدام تقنية تحديد المعايير والثوابت لحل بعض المسائل في حسبان التغاير. تقنية تحديد المعايير والثوابت والمعتمدعلى متعددات متعدد الحدود تم تقديمه لاختزال مسالة التغاير الى مسالة البرمجة التربيعية. اعطيت بعض الامثلة لتوضيح صحة الطريقة.

INTRODUCTION

The calculus of variations is a branch of mathematical analysis that studies extrema and critical points of functionals (or energies). Functional minimization problems known as variational problems appear in engineering and science where minimization of functionals, such as Lagrangian, strain, potential, total energy, give the lows governing the systems behavior [1].

Some popular methods for solving variational problems are direct methods. A fast numerical method for solving calculus of variation problems was given by [1], the direct method of Ritz, Walsh functions [2], Laguerre series [3], shifted Legender polynomial series [4], shifted Chebychev series [5], Fourier series [6] and nonclassical parameterization [7] have been applied to solve variational problems.

In this work, we propose a direct method using parameterization technique with Laguerre and Hermite polynomials for finding the extremal of some variational problems. The application of the method will convert the variational problem to quadratic

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programming problem, then using Lagrange multiplier technique to find the unknown parameters.

Orthogonal Polynomials

Special orthogonal polynomials began appearing in mathematics before the significance of such a concept became clear. For instance, Laplace used Hermite polynomials in his studies in probability while Legendre and Laplace utilized Legender polynomials in celestial mechanics. Some properties of Hermite and Laguerre are given because these are the most extensively studied and here the longest history [8].

Hermite Polynomials and Their Properties

Hermite polynomials and its applications have been studied for long and still attract attention. One can refer a long and list of books and journals for advanced knowledge of Hermite polynomials and its extension, for example [9], for books, and [10],[11]and[12] for journals.

Definition [13]

The nth Hermite polynomials denoted by $H_n(t)$ are given by

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n e^{-t}}{dt^n} \qquad n = 0,1,2$$

also $H_n(t)$ can be expressed as

$$H_n(t) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \frac{n!}{k!(n-2k)!}(2t)^{n-2k}$$ and $\left\lfloor \frac{n}{2} \right\rfloor$ denotes the largest integer $\leq \frac{n}{2}$

The Hermite polynomials have many important properties are

- The initial values

$$H_{2n}(0) = (-1)^n \frac{2n!}{n!}, \qquad \dot{H}_{2n}(0) = 0$$

$$H_{2n+1}(0) = 0 \qquad \dot{H}_{2n+1}(0) = (-1)^n \frac{(2n+2)!}{(n+1)!}$$

- Product property

$$H_n(t)H_m(t) = \sum_{k=0}^{\min(n,m)} \binom{m}{k}\binom{n}{k} H_{-2k+m+n}(t) 2^k k!$$

$$H_n(t)H_m(t) = n!m! \sum_{k=0}^{\min(n,m)} \frac{2^k H_{-2k+m+n}(t)}{k!(n-k)!(m-k)!}$$

- Integration property

$$\int_0^x H_n(t)dt = \frac{1}{2(n+1)}(H_{n+1}(x) - H_{n+1}(0))$$

- Differentiation property

$$\frac{dH_n(t)}{dt} = 2nH_{n-1}(t)$$

$$\frac{d^2H_n(t)}{dt^2} = 4n(n-1)H_{n-2}(t)$$

- Operational matrix of derivative D_H:

$$D_H = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 2n & 0 \end{pmatrix}$$

Laguerre Polynomials and Their Properties

The Laguerre polynomials were introduced by Edmond Laguerre more than 150 years ago. Many applications of them on various problems in mathematics, physics and electrotechnics were found. The Laguerre polynomials are orthogonal polynomials with respect to the weighting function e^{-t} on the half-line $[0,\infty)$ they are denoted by the letter "L" with the order as subscript and normalized by the condition that the coefficient of the higest order term of L_n is $(-1)^n / n!$, [14].

Definition [15]

The nth Laguerre polynomials are given by the formula

$$L_n(t) = e^t \frac{d^n}{dt^n}(t^n e^{-t})$$

also $L_n(t)$ can be expressed as

$$L_n(t) = \sum_{k=0}^{n} (-1)^k \frac{(n!)^2}{(k!)^2 (n-k)!} (t)^k$$

The Laguerre polynomials have many important properties, are

- The initial values

$L_n(0) = n!$

$$\dot{L}_n(0) = -n \times n!$$

- Integration property

$$\int_0^x L_n(t)dt = L_n - \frac{L_{n+1}(x)}{n+1}$$

- Differentiation property

$$\dot{L}_n(t) = -n!\sum_{i=0}^{n-1}\frac{L_i(t)}{i!} \quad n \geq 1$$

$$L_n^{\ m}(t) = (-1)^m \frac{n!}{(m-1)!}\sum_{i=0}^{n-m}\frac{(n-(i+1))!}{i!(n-m-i)!}L_i(t)$$

- Operational matrix of derivative D_L :

- $$D_L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -2 & -2 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -n! & -n! & \cdots & \cdots & \frac{-n!}{(n-1)!} & 0 \end{pmatrix}$$

Lagrange Methods and Quadratic Programming

Lagrange methods [16]

The Lagrange methods for dealing with constrained optimization problem are based on solving the Lagrange first-order necessary conditions.
In particular, for solving the problem with equality constraints only:

Minimize f(x)

$$g_j = 0 \qquad j = 1,2,...,m < n$$

the Lagrangian function is follows:

$$L(x,\lambda) = f(x) + \sum_{j=1}^{m}\lambda_j g_j(x)$$

$$= f(x) + \lambda^T g(x)$$

$$\lambda^T = (\lambda_1, \lambda_2, ..., \lambda_m) \text{ and } \quad g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_m(x) \end{pmatrix}$$

where λ is an ($m \times 1$)vector of Lagrange multipliers.

In general, we can set the partial derivative to zero to find the minimum:

$$\frac{\partial L}{\partial x_i}(x,\lambda)=0 \qquad i=1,\dots,n$$

$$\frac{\partial L}{\partial \lambda_j}(x,\lambda)=0 \qquad j=1,\dots,m$$

where x^* is the minimum solution and λ^* is the set of the associated Lagrange multipliers.

Quadratic programming [16]

An important special case is when the target function f is quadratic and the constraints are linear:

Minimize $(\frac{1}{2})x^T Qx + x^T c$

subject Ax=b

Q is asymmetric matrix, x^T is transpose of x and the Lagrange necessary conditions become:

$$L(x,\lambda)=\frac{1}{2}x^T Qx + x^T c + \lambda^T (Ax-b)$$

$$\frac{\partial L}{\partial x}=Qx+c+\lambda^T A$$

$$\frac{\partial L}{\partial \lambda}=Ax-b$$

Since $(\lambda^T A = A^T \lambda)$ therefore

$$Qx+c+A^T\lambda=0$$

$$Ax-b=0$$

If Q is nonsingular then the solution becomes:

$$x=-Q^{-1}C+Q^{-1}A^T(AQ^{-1}A^T)^{-1}[AQ^{-1}c+b]$$

$$\lambda=-(AQ^{-1}A^T)^{-1}[AQ^{-1}c+b]$$

Direct Method Using Hermite Polynomials

The simplest form of a variational problem can be considered as finding the extremum of the value of the functional [7]:

$$J[x(t)]=\int_{t_0}^{t_1} F[x(t),\dot{x}(t),t]dt \qquad \dots(1)$$

To find the extreme value of J, the boundary points of the admissible curves are known in the following form

$$x(t_0)=\alpha, \quad x(t_1)=\beta \qquad \dots(2)$$

Suppose the variable $x_i(t)$ can be expressed approximately as

$$x_i(t)=\sum_{j=0}^{N}a_{ij}H_j(t)=a_i^{\,T}H(t) \qquad i=1,2,\ldots,n, \qquad j=0,1,\ldots,N \qquad \ldots(3)$$

where $a_i=[a_{i0},a_{i1},\ldots,a_{iN}]^T, i=1,2,\ldots,n$ are $(N+1)\times 1$ vector of unknown parameters with property that $a_{ij}=x_i(t_j)$ and $H(t)=[H_0,H_1,\ldots,H_N]^T$ is $(N+1)\times 1$ vector of Herimte polynomial $H(t)$.

Differentiate eq.(3), yields:

$$\dot{x}_i(t)=a_i^{\,T}\dot{H}(t),=a_i^{\,T}D_H H(t) \qquad i=1,2,\ldots,n \qquad \ldots(4)$$

Where D_H is the operational matrix of derivative for Hermite polynomials
Substituting eqs.(3)and(4) into eq.(1), the functional J becomes a nonlinear mathematical programming problem of unknown parameters a_{ij} , where i=1,2,…,n, and j=0,1,…,N . Hence to find the extremum of J, we solve

$$J(x)=\frac{1}{2}a^T\mathbf{H}a+c^T a \qquad \ldots (5)$$

where $\mathbf{H}=\int_{t_0}^{t_1}[H(t),D_H H(t)]dt$

$$c^T=\int_{t_0}^{t_1}[t,H(t),D_H H(t)]dt$$

eq.(3) and the boundary conditions in eq.(2) imply

$$x(t_0)=a^T H(t_0)=\alpha,\ x(t_1)=a^T H(t_1)=\beta \qquad \ldots(6)$$

The quadratic programming problem in eqs.(5)and(6) can be rewritten as follows :

minimize $J(x)=\frac{1}{2}a^T\mathbf{H}a+c^T a$

Subject to

$$Fa-b=0,$$

where

$$F=\begin{pmatrix}H^T(t_0)\\ H^T(t_1)\end{pmatrix}, \qquad b=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

The optimal values of unknown parameters, a^* , can be obtained easily using Lagrange multiplier technique as

$$a^*=-\mathbf{H}^{-1}c+\mathbf{H}^{-1}F^T(F\mathbf{H}^{-1}F^T)^{-1}(F\mathbf{H}^{-1}c+b)$$

Then substituting a^* in eq.(3) to get the solution, we establish the detailed procedure via some problems
The same procedure can be used with Laguerre polynomials and its operational matrix of derivative.

Numerical Examples

First order functional extremal with two fixed boundary conditions

Consider the problem of finding the minimum of the time-varying functional

$$J(x) = \int_0^1 [x^2 + 2t\dot{x} + \dot{x}^2]dt \qquad \ldots(7)$$

with boundary conditions x(0)=2, x(1)= $e^1 + 1$, the exact solution is $e^t + 1$
for solving this problem by the Hermite polynomials, let

$$x(t) = a^T H = a_0H_0 + a_1H_1 + a_2H_2 + a_3H_3 + a_4H_4 \qquad \ldots(8)$$

Differentiate eq.(8) to get

$$\dot{x}(t) = a^T D_H H(t) \qquad \ldots(9)$$

where

$$D_H = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \end{pmatrix}$$

Also, in view of boundary conditions, we have

$$x(0) = a_0 - 2a_2 + 12a_4 = 2 \qquad \ldots(10)$$

$$x(1) = a_0 + 2a_1 + 2a_2 - 4a_3 - 20a_4 = e^1 + 1 \qquad \ldots(11)$$

Now, by using eqs.(8) and (9), the cost function J may be rewritten as

$$J(x) = \int_0^1 [a^T H(t)H^T(t)a + 2a^T tD_H H(t) + a^T D_H H(t)(D_H H)^T(t)a]dt \qquad \ldots(12)$$

Eqs(10-12) can be simplified to quadratic programming problem

$$\text{minimize} \quad J(x) = \frac{1}{2}a^T \mathbf{H}a + c^T a$$

Subject to

$$Fa - b = 0,$$

where

$$F = \begin{pmatrix} H^T(0) \\ H^T(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 & 0 & 12 \\ 1 & 2 & 2 & -4 & -20 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ e^1 + 1 \end{pmatrix}$$

$$c^T = 2\int_0^1 t(D_H H)^T(t)dt = \begin{pmatrix} 0 & 2 & \frac{16}{3} & 0 & -\frac{192}{5} \end{pmatrix}$$

$$H = 2\int_0^1 [H(t)H^T(t) + D_H H(t)(D_H H)^T(t)]dt$$

$$= \begin{pmatrix} 2 & 2 & \frac{-4}{3} & -8 & -\frac{8}{5} \\ 2 & \frac{32}{3} & 16 & \frac{-128}{5} & \frac{-424}{3} \\ \frac{-4}{3} & 16 & \frac{232}{5} & \frac{8}{3} & \frac{-11568}{35} \\ -8 & \frac{-128}{5} & \frac{8}{3} & \frac{6016}{35} & 160 \\ \frac{-8}{5} & \frac{-424}{3} & \frac{-1168}{35} & 160 & \frac{164576}{63} \end{pmatrix}$$

The optimal values of unknown parameters a_i, $i = 0,1,2,3,4$ can be obtained easily using Lagrange multiplier technique as

$$a_0 = 2.3068223$$
$$a_1 = 0.60484444$$
$$a_2 = 0.17953216$$
$$a_3 = 0.01755142$$
$$a_4 = 0.00435349$$

The approximate solution is : x(t)=

$$2.3068223H_0 + 0.6048444H_1 + 0.179532168H_2 + 0.01755142H_3 + 0.00435349H_4$$

Table (1) shows the approximate solution obtained by using Hermite polynomials with M=4 and the exact solution.

Table (1) comparison between exact and approximate solution.

t	exact	Hermite M=4	Absolute error
0	2	2	0
0.1	2.1051709	2.1015461	0.0000248
0.2	2.2214027	2.2214155	0.0000128
0.3	2.3498588	2.3499013	0.0000425
0.4	2.4918246	2.4918639	0.0000393
0.5	2.6487212	2.6487310	0.0000098
0.6	2.8221188	2.8220972	0.00000216
0.7	3.0137527	3.0137245	0.00000282
0.8	3.2255409	3.2255421	0.0000012
0.9	3.4596031	3.4596409	0.0000378
1	3.7183000	3.7183000	0.0000000

This problem also solved by using Laguerre polynomial to fined approximate solution with M=4, the same result are obtained.

First order functional extremal with a fixed and a moving boundary conditions

We consider the same functional extermal of eq.(7) but with unspecified $x(1)$, namely $x(0)=2$, $x(1)=\text{unspecified}$

Another condition may be found from $F[t,x(t),\dot{x}(t)]$

$F_{\dot{x}}|_{t=1}=0$, that is $2\dot{x}+2t=0$ at $t=1$ or $\dot{x}(1)=-1$

In this case, the exact solution is $x(t)=c_1e^{t}+c_2e^{-t}+1$, where $c_1=\dfrac{e^{-1}-1}{e^{1}+e^{-1}}$, $c_2=\dfrac{1+e^{1}}{e^{1}+e^{-1}}$

The equation (7) become J(x) = $\int_0^1 x^2+2t\dot{x}+\dot{x}^2dt$ with boundary condition $x(0)=2$, $\dot{x}(1)=-1$, for solving this problem by Laguerre polynomials, let

$$x(t)=a^TL=a_0L_0+a_1L_1+a_2L_2+a_3L_3+a_4L_4 \qquad \ldots(13)$$

Differentiate eq.(13) to get

$$\dot{x}(t)=a^TD_LL(t) \qquad \ldots(14)$$

where

$$D_L=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -2 & -2 & 0 & 0 & 0 \\ -6 & -6 & -3 & 0 & 0 \\ -24 & -24 & -12 & -4 & 0 \end{pmatrix}$$

Also, in view of boundary conditions, we have

$$x(0)=a_0-2a_2+6a_3+24a_4=2 \qquad \ldots(15)$$

$$\dot{x}(1)=-a_1-2a_2-3a_3+4a_4=-1 \qquad \ldots(16)$$

Now, by using eqs.(13) and (14), the cost function J may be rewritten as

$$J(x)=\int_0^1[a^TL(t)L^T(t)a+2a^TtD_LL(t)+a^TD_LL(t)(D_LL)^T(t)a]dt \qquad \ldots(17)$$

Eqs(15-17) can be simplified to quadratic programming problem

$$\text{minimize} \quad J(x)=\frac{1}{2}a^T\mathbf{H}a+c^Ta$$

subject to

$$Fa-b=0,$$

where

$$F=\begin{pmatrix} L^T(0) \\ D_L L^T(1) \end{pmatrix}=\begin{pmatrix} 1 & 0 & -2 & 6 & 24 \\ 0 & -1 & -2 & -3 & 4 \end{pmatrix}, \qquad b=\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$c^T=2\int_0^1 t(D_L L)^T(t)dt=\begin{pmatrix} 0 & -1 & \frac{-8}{3} & \frac{-15}{2} & \frac{-112}{5} \end{pmatrix}$$

$$H=2\int_0^1 [L(t)L^T(t)+D_L L(t)(D_L L)^T(t)]dt$$

$$=\begin{pmatrix} 2 & 1 & \frac{2}{3} & \frac{-1}{2} & \frac{-38}{5} \\ 1 & \frac{8}{3} & \frac{41}{6} & \frac{107}{5} & \frac{1207}{15} \\ \frac{2}{3} & \frac{41}{6} & \frac{102}{5} & \frac{1048}{15} & \frac{29822}{105} \\ \frac{-1}{2} & \frac{107}{5} & \frac{1048}{15} & \frac{8914}{35} & \frac{153789}{140} \\ \frac{-38}{5} & \frac{1207}{15} & \frac{29822}{105} & \frac{153789}{140} & \frac{315446}{63} \end{pmatrix}$$

The optimal values of unknown parameters a_i, $i=0,1,2,3,4$ can be obtained easily using Lagrange multiplier technique as

$a_0=0.66805726$
$a_1=1.79719137$
$a_2=-0.3210646$
$a_3=-0.0313954$
$a_4=0.01521889$

The approximate solution is :
x(t)=

$$=0.66805726L_0+1.79719137L_1-0.3210646L_2-0.0313954L_3+0.01521889L_4$$

Table (2) shows the approximate solution obtained by using Laguerre polynomials with M=4 and the exact solution.

Table (2) comparison between exact and approximate solution.

t	exact	Laguerre M=4	Absolute error
0	2	2	0
0.1	1.8638042	1.8638279	0.0000237
0.2	1.7362537	1.7362472	0.00000065
0.3	1.6160719	1.6160401	0.0000318
0.4	1.5020560	1.5020252	0.0000308
0.5	1.3930647	1.3930578	0.00000698
0.6	1.2880074	1.2880294	0.000022
0.7	1.1858326	1.1858685	0.0000359
0.8	1.0855177	1.0855397	0.000022
0.9	0.9860586	0.9860441	0.0000145
1	0.8864011	0.8864203	0.0000192

This problem also solved by using Hermite polynomial to fined approximate solution with M=4, the same result are obtained.

Conclusion

Direct parameterization technique was employed for finding the solution of first order functional extremal with fixed and moving boundary conditions using Laguerre and Hermite polynomials as abases functions.

The presented methods with Laguerre and Hermite basis functions were applied to solve variational problems and the same results were obtained.

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