Developing a Mathematical Method for Controlling the Generation of Cubic Spline Curve based on Fixed Data Points, Variable Guide Points and Weighting Factors

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ABSTRACT:
The cubic spline interpolation method is used to design the developed curve. The method enables the designer to change the shape of the curve to the desired one without changing the original data points. That done by changing the value of the parameter \( t \) depending on defined guide points and weighting factors that assigned secretly by the designer, so that not easily recreate the same design by anyone else even he knows the original data points that used to design the curve. These curves can be used in many fields such as banknote design, secure printed documents, the design of cars and airplanes structures, and Ornament. This paper modifies a mathematical technique to design complex curves, which are not easy to recreate. The results of this research had compared with other related methods since the improved models of this research is used the negative values of the guide to compute the parametric rather than using the only positive ones; the efficiency of the modified technique had proved by several comparative examples.

Keywords: Cubic spline, negative parametric, weighting factor, Area Parameterization.

الخلاصة:
الطريقة تمكن المصمم من تغيير شكل المنحنى إلى الشكل المرغوب فيه من دون الحاجة إلى تغيير نقاط البيانات الأصلية للمنحنى. يتم ذلك من خلال تغيير قيمة المتغير (\( t \)) بالاعتماد على نقاط الاتصال ومعاملات الوزن توضع بسرية تامة من قبل المصمم، بحيث ليس من السهل لأي شخص آخر تقدير المنحنى حتى في حالة كشف نقاط البيانات الأصلية المستخدمة في تصميم المنحنى. هذه المنحنى يمكن أن تستخدم في الكثير من المجالات ومنها تصميم العملات النقدية، تصميم هياكل السيارات، تأمين الوثائق المطبوعة، تصميم هياكل السيارات...
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INTRODUCTION

The simplest kind of cubic spline consists of one cubic polynomial, which has degree at most three, with prescribed values and prescribed slopes at two points. A general cubic spline consists of several pieces of cubic polynomials, joined to form a twice-differentiable function. Thus, the inclusion of a few exercises with cubic splines in calculus reinforces the idea that one function need not consist of a single formula, but may involve several algebraic formulae and logical tests, that some problems involve not curve sketching, but building a function subject to geometric specifications, and that such functions and problems have practical applications [1, 2].

Many previous researches discussed in cubic spline area; Rahma [1] provided the parameterization method for controlling the design by using only one guide point. This research had developed the area parameterization method for giving higher flexibility to generate and control the design by calculating the triangular area between each data point and the defined guide points with or without the use of the weighting factors ($W_i$).

This paper presents a Modified mathematical method for controlling curves’ generation by using guide points and weighting factors with the constraint of cubic spline. The method proposed for using two guide points, three guide points, two guide points with two weighting factors, three guide points with three weighting factors, the values of the parameter ($t_i$) had computed from the guide points and/or the weighting factors in both positive and negative cases. Spline curves are very important, it can be used in the design of cars, ships, airplanes, banknotes, passports, securities, university certificate, and other things, which are of high impotence. The curve generation algorithm should allow the designer to easily design a complex curves. For such purpose, this research had been made.

The rest of this paper is organized as follows: section 2 provides a brief description about cubic spline Preliminaries and, section 3 describes the developed area parameterization method in four cases, section 4 gives a brief description about area parameterization model, and section 5 summarizes the conclusions about the results of this research.

Cubic Spline Background

Spline theory is simple. Over $n$ intervals, the routine fits $n$ equations subject to the boundary conditions of $n+1$ data points. The derivations of Lilley and Wheatly are used. The derivation assumes a functional form for the curve fit. This equation (Eq. (1)) form is simplified and then solved for the curve fit equation [3, 4, 5].

$$y = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i$$

Where
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the \(x, y\) represent the points location on axis; The assumed form for the cubic polynomial curve fit for each segment are the \((a,b,c,d)\)coefficients.

Where

\[
\begin{bmatrix}
2(h_1 + h_2) & h_2 & \cdots & h_2 \\
h_2 & 2(h_2 + h_3) & \cdots & h_3 \\
\vdots & \vdots & \ddots & \vdots \\
h_{n-2} & h_{n-2} & \cdots & 2(h_{n-2} + h_{n-1})
\end{bmatrix}
\begin{bmatrix}
S_1 \\
S_2 \\
\vdots \\
S_{n-1}
\end{bmatrix}
= 6
\begin{bmatrix}
y_3 - y_2 \\
y_2 - y_1 \\
\vdots \\
y_n - y_{n-1} \\
y_{n-1} - y_{n-2}
\end{bmatrix}
\]

the spacing between successive data points is shown in Eq. (2) below:

\[h_i = x_{i+1} - x_i \quad \ldots (2)\]

The cubic spline constrains the function value, 1\(^{st}\) derivative \(\frac{dy}{dx}\) (x) and 2\(^{nd}\) derivative \(\frac{d^2y}{dx^2}\) (x). The routine must ensure that \(y(x), \frac{dy}{dx}(x)\) and \(\frac{d^2y}{dx^2}(x)\) are equal at the interior node points for adjacent segments.

Substituting a variable \(S\) for the polynomial’s second derivative reduces the number of equations from \(a, b, c, d\) for each segment to only \(S\) for each segment.

For the \(i^{th}\) segment, the \(S\) governing equation shown below in Eq. (3),

\[h_{i-1} S_{i-1} + (2h_i + 2h_{i+1})S_i + h_i S_{i+1} = 6 \left( \frac{y_{i+1} - y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}} \right) \quad \ldots (3)\]

In matrix form, the governing equations reduce to a tri-diagonal form.

\(S_1\) and \(S_n\) are zero for the natural spline boundary condition. If different boundary conditions are needed, the appropriate changes can be made to the governing equations.

Finally, the cubic spline properties are found by substituting into the equations shown in Eq. (4), Eq. (5), Eq. (6), and Eq. (7) . These \(a, b, c\) and \(d\) values correspond to the polynomial definition for each segment. [3, 6, 7]

\[a_i = (S_{i+1} - S_i) / 6h_i \quad \ldots (4)\]

\[b_i = S_i / 2 \quad \ldots (5)\]

\[c_i = \frac{y_{i+1} - y_i}{h_i} - \frac{2h_i S_i + h_{i+1} S_{i+1}}{6} \quad \ldots (6)\]

\[d_i = y_i \quad \ldots (7)\]

Area Parameterization Model
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The control of the cubic spline’s shape can be done based on the area parameterization (Fig 1) by choosing a point \( G = (X_G, Y_G) \) called the guide of parameterization [1], then we define the values of \( t_i \) to be as shown in Eq. (8)

\[
t_0 = 0
\]

\[
t_i = t_{i-1} + |\delta_i|/D \text{ for } i=1,\ldots,N
\]

where

\[
\delta_i = \frac{1}{2} \left| \begin{array}{cc} X_i - X_G & Y_i - Y_G \\ X_{i-1} - X_G & Y_{i-1} - Y_G \end{array} \right| \text{ for } i=1, 2, \ldots, N
\]

And \( D = \sum_{i=1}^{N} |\delta_i| \) for \( i=1, 2, \ldots, N \)

**Figure (1): Guide of Parameterization**

**Developed Area Parameterization Model:**

The area parameterization model in [1] has been tested and developed in many perspectives:

1. After programming and carefully testing the mathematical model mentioned in [1], this paper added another case which is using negative \((G_x, G_y)\) values to change the shape of the curve without changing the original data points values. By computing the parameter \((t_i)\) values from negative guide points \((G_x, G_y)\) rather than using only the positive ones as mentioned in [1]. **Fig 2** shows the result of drawing 5 data points using cubic spline interpolation with area parameterization model and one guide point with positive \(G(300,250)\), negative \(X, G(-300,250)\), negative \(Y, G(300,-250)\), and negative \(X,Y G(-300,-250)\) values.
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II. The area parameterization model [1] has modified by using more than one guide point to control the design to the desired shape without changing the original data points. The mathematical model developed on two phases as described below:

Case 1: At the first case two guide points \( G_1 = (G_1X, G_1Y) \) and \( G_2 = (G_2X, G_2Y) \) had added to control the design by calculating the parameter \( t_i \) values from the calculation of the triangular distances \( \delta_i \) between each guide point and all the data points \( (X_i, Y_i) \) as shown in the following mathematical models:

Model 1: Two Guide Points for area parameterization

\[
t_0 = 0
\]

\[
t_i = t_{i-1} + \left| \frac{\delta_{1i} + \delta_{2i}}{2} \right| \left( \frac{D_1 + D_2}{2} \right) \quad \text{for } i=1,2,\ldots,N \tag{11}
\]

such that:

\[
\delta_{1i} = \frac{1}{2} \left| \begin{array}{cc}
X_i - G_1X & Y_i - G_1Y \\
X_{i-1} - G_1X & Y_{i-1} - G_1Y
\end{array} \right| \quad \text{for } i=1,2,\ldots,N \tag{12}
\]

\[
D_1 = \sum_{i=1}^{N} |\delta_{1i}| \quad \text{for } i=1,2,\ldots,N \tag{13}
\]

\[
\delta_{2i} = \frac{1}{2} \left| \begin{array}{cc}
X_i - G_2X & Y_i - G_2Y \\
X_{i-1} - G_2X & Y_{i-1} - G_2Y
\end{array} \right| \quad \text{for } i=1,2,\ldots,N \tag{14}
\]

Figure (2): controlling the design of the curve using one guide point with positive and negative values \([A.G (300,250), B. G(-300,250), C.G(300,-250), D. G(-300,-250)]\)
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\[ D_2 = \sum_{i=1}^{n} |\delta_{2i}| \quad \text{for} \quad i=1,2,\ldots,N \quad \text{…. (15)} \]

\( N \) the number of the data points

The model gives a higher flexibility than the case of using only one guide point to design the curve, the designer has more options to change the curve’s shape in more than one method as described without changing the original data points.

1. Changing the design by changing one of the guide point’s location or both of them in the positive or negative values or both of them, in the case of using negative values, the curve shape will be opposite to the other side with the same design. Fig3 illustrate the case of using two guide points and five data points with mathematical model described in case1.

2. Changing the design without changing the location of the guide points nor changing the location of the original data points by adding a (\( W_1 \)) with each guide point, the \( W_i \) represents the percentage effect of the specified guide on the design of the curve. By the experiment, noticed that the design can be controlled much easier when the summation of the weight factors equal to 1 (\( \sum_{i=1}^{2} W_i = 1 \)). The other case (when\( \sum_{i=1}^{2} W_i \neq 1 \)) also changed the design, but with less flexibility. As shown in the following mathematical model

**Model 2**: Two guide Points (\( G_1,G_2 \))for area parameterization with 2 Weight factors percentages (\( W_1 \& W_2 \)) of impact on each guide point. Parameters of this model are shown below in Eq. (16), Eq. (17), Eq. (18), Eq. (19), and Eq. (20).

\[ t_i = t_{i-1} + \left( \frac{(\delta_{1i} + W_1) + (\delta_{2i} + W_2)}{2} \right) \times \left( \frac{D_1 + D_2}{2} \right) \quad \text{for} \quad i=1,2,\ldots,N \quad \text{…. (16)} \]

such that:

\[ \delta_{1i} = \frac{1}{2} \begin{vmatrix} X_i - G_1 X & Y_i - G_1 Y \\ X_{i-1} - G_1 X & Y_{i-1} - G_1 Y \end{vmatrix} \quad \text{for} \quad i=1,2,\ldots,N \quad \text{…. (17)} \]

\[ D_1 = \sum_{i=1}^{n} |\delta_{1i}| \quad \text{for} \quad i=1,2,\ldots,N \quad \text{…. (18)} \]

\[ \delta_{2i} = \frac{1}{2} \begin{vmatrix} X_i - G_2 X & Y_i - G_2 Y \\ X_{i-1} - G_2 X & Y_{i-1} - G_2 Y \end{vmatrix} \quad \text{for} \quad i=1,2,\ldots,N \quad \text{…. (19)} \]

\[ D_2 = \sum_{i=1}^{n} |\delta_{2i}| \quad \text{for} \quad i=1,2,\ldots,N \quad \text{…. (20)} \]

Where

\( N \) performs number of the data points.

The value of \( W_i \) can be negative or positive integer or float numbers.

The mathematical model of **model2** allows more control over designing than the case of using two guides without the use of \( W_i \).

The design can be controlled with many options by changing one of the guide points, the weight factors, the data points, or all of them together with positive and negative values as shown in the Fig4, Fig5, Fig6, and Fig7.
Case 2: At the second case three guide points $G_1=(G_1X,G_1Y), G_2= (G_2X,G_2Y)$ and $G_3= (G_3X,G_3Y)$ had added to control the design by calculating the values of the parameter $t_i$ from the calculation of the triangular distances $\delta_i$ between each guide point and all the data points $(X_i,Y_i)$ as shown in the mathematical models.

Model 3: Three Guide Points for area parameterization, equations below (Eq. (21), Eq. (22), Eq. (23), Eq. (24), Eq. (25), Eq. (26), and Eq. (24)) performs the mathematical presentation of this model.

\[ t_i = t_{i-1} + \left| \frac{\delta_{3i} + \delta_{2i} + \delta_{1i}}{3} \right| \times \left( \frac{D_1 + D_2 + D_3}{3} \right) \quad \text{for } i=1,2,...,N \]  

such that:

\[ \delta_{1i} = \frac{1}{2} \times \begin{vmatrix} X_i - G_1X & Y_i - G_1Y \\ X_{i-1} - G_1X & Y_{i-1} - G_1Y \end{vmatrix} \quad \text{for } i=1,2,...,N \]  

\[ D_1 = \sum_{i=1}^{N} \left| \delta_{1i} \right| \quad \text{for } i=1,2,...,N \]  

\[ \delta_{2i} = \frac{1}{2} \times \begin{vmatrix} X_i - G_2X & Y_i - G_2Y \\ X_{i-1} - G_2X & Y_{i-1} - G_2Y \end{vmatrix} \quad \text{for } i=1,2,...,N \]  

\[ D_2 = \sum_{i=1}^{N} \left| \delta_{2i} \right| \quad \text{for } i=1,2,...,N \]  

\[ \delta_{3i} = \frac{1}{2} \times \begin{vmatrix} X_i - G_3X & Y_i - G_3Y \\ X_{i-1} - G_3X & Y_{i-1} - G_3Y \end{vmatrix} \quad \text{for } i=1,2,...,N \]  

\[ D_3 = \sum_{i=1}^{N} \left| \delta_{3i} \right| \quad \text{for } i=1,2,...,N \]

Where

$N$ is the number of the control points.

The model allows higher degree of control over the design than the previous cases of using one or two guide's points.

In addition, the designer can use positive and negative values to control the design and modify the $t_i$ value without changing the data points, Fig8 and Fig9 show a few samples designed using the case3 mathematical model.

Also, here the designer can control the design without changing the data point nor the values of the three guide points by adding $(W_i)$ with each guide point as shown in the mathematical model of Model4. By the experiment, noticed that the design can be controlled much easier when the summation of the weight factors equal to 1 ($\sum_{i=1}^{3} W_i = 1$). While the other case (when $\sum_{i=1}^{3} W_i \neq 1$) also changed the design, but with less flexibility.

Model 4: Three Guide Points $(G_1, G_2, G_3)$ for area parameterization with three weight factors $(W_1, W_2, W_3)$ of impact on each guide point. The mathematical model is shown below in the following equations:

\[ t_0 = 0 \]
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\[ t_i = t_{i-1} + \left( \frac{(\delta_{1i} + \delta_{2i} + \delta_{3i})}{3} \right) * \left( \frac{D_1 + D_2 + D_3}{3} \right) \text{ for } i=1,2,..,N \quad \ldots (28) \]

such that:

\[ \delta_{1i} = \frac{1}{2} \cdot \begin{vmatrix} X_i - G_1 X & Y_i - G_1 Y \\ X_{i-1} - G_1 X & Y_{i-1} - G_1 Y \end{vmatrix} \text{ for } i=1,2,..,N \quad \ldots (29) \]

\[ D_1 = \sum_{i=1}^{n} \left| \delta_{1i} \right| \text{ for } i=1,2,..,N \quad \ldots (30) \]

\[ \delta_{2i} = \frac{1}{2} \cdot \begin{vmatrix} X_i - G_2 X & Y_i - G_2 Y \\ X_{i-1} - G_2 X & Y_{i-1} - G_2 Y \end{vmatrix} \text{ for } i=1,2,..,N \quad \ldots (31) \]

\[ D_2 = \sum_{i=1}^{n} \left| \delta_{2i} \right| \text{ for } i=1,2,..,N \quad \ldots (32) \]

\[ \delta_{3i} = \frac{1}{2} \cdot \begin{vmatrix} X_i - G_3 X & Y_i - G_3 Y \\ X_{i-1} - G_3 X & Y_{i-1} - G_3 Y \end{vmatrix} \text{ for } i=1,2,..,N \quad \ldots (33) \]

\[ D_3 = \sum_{i=1}^{n} \left| \delta_{3i} \right| \text{ for } i=1,2,..,N \quad \ldots (34) \]

Where

\( N \) is the number of the control points

With the Mathematical Model4, the designer has the largest number of design options; he can design the curve by changing one of the parameters or all of them at the same step with negative or positive values until he gets the desired design. Fig10 and Fig11 show a few samples designed using the mathematical Model4.

Figure (3): shows the degree of change in the curve shape with different weighting factors on the same data points and guide points.
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Figure (4): The designs with and without weighting factors here is a negative weighting factors values.

Figure (3): shows the degree of change in the curve shape with different weighting factors on the same data points and guide points.

G₁=(100,200) G₂=(-320,-400) with \( \hat{W}_1 = -0.75 \) \( \hat{W}_2 = 0.75 \)

G₁=(100,200) G₂=(-320,-400) with \( \hat{W}_1 = 0.1 \) \( \hat{W}_2 = 0.9 \)
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Figure (5): The update of the design by changing the weighting factors values.

\[ G_1 = (100, 200) \quad G_2 = (-320, -400) \quad \text{with} \quad \hat{W}_1 = 2, \ \hat{W}_2 = -1 \]

\[ G_1 = (100, 200) \quad G_2 = (-320, -400) \quad \text{with} \quad \hat{W}_1 = -99, \ \hat{W}_2 = 100 \]

Figure (6): (a, b) The case of changing the guide point without changing the weighting factors and the data point values.
Figure (7): (a) Design curve using five data points and three guide points $G_1=(180,-200)$, $G_2=(-290,50)$, $G_3=(490,150)$, (b) Design curve using five data points and three guide points $G_1=(180,200)$, $G_2=(290,50)$, $G_3=(-490,-150)$, (c) Design curve using five data points and three guide points $G_1=(180,200)$, $G_2=(290,50)$, $G_3=(490,150)$, (d) Design curve using five data points and three guide points $G_1=(180,200)$, $G_2=(-290,-50)$, $G_3=(490,150)$. 
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Figure (8): (a) Design curve using five data points and three guide points $G_1=(180,200)$, $G_2=(-290,50)$, $G_3=(490,150)$, (b) Design curve using five data points and three guide points $G_1=(92,200)$, $G_2=(290,250)$, $G_3=(390,150)$, (c) Design curve using five data points and three guide points $G_1=(-180,-200)$, $G_2=(290,50)$, $G_3=(-490,-150)$, (d) Design curve using five data points and three guide points $G_1=(180,200)$, $G_2=(290,250)$, $G_3=(390,150)$. 
Figure (9): (a) Design a curve using five data points and three guide points plus three weight factors \( G_1=(180,200), G_2=(-290,-50), G_3=(490,150), \tilde{\omega}_1=-0.4, \tilde{\omega}_2=0.8, \tilde{\omega}_3=0.6, \) (b) Design a curve using five data points and three guide points plus three weight factors \( G_1=(180,200), G_2=(-290,-50), G_3=(490,150), \tilde{\omega}_3=0.8, \tilde{\omega}_2=0.4 \tilde{\omega}_3=0.6, \) (c) Design a curve using five data points and three guide points plus three weight factors \( G_1=(180,200), G_2=(-290,-50), G_3=(490,150), \tilde{\omega}_1=9, \tilde{\omega}_2=-6, \tilde{\omega}_3=-2, \) (d) Design a curve using five data points and three guide points plus three weight factors \( G_1=(180,200), G_2=(-290,-50), G_3=(490,150), \tilde{\omega}_1=-9, \tilde{\omega}_2=-4, \tilde{\omega}_3=-2.\)
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Figure (10): (a) Design a curve using five data points and three guide points plus three weight factors \( G_1 = (180, 200), G_2 = (-290, -50), G_3 = (490, 150), W_1 = -25, W_2 = -25, W_3 = 51 \), (b) Design a curve using five data points and three guide points plus three weight factors \( G_1 = (180, 200), G_2 = (-290, -50), G_3 = (490, 150), W_1 = -24, W_2 = -26, W_3 = 51 \), (c) Design a curve using five data points and three guide points plus three weight factors \( G_1 = (180, 200), G_2 = (-290, -50), G_3 = (490, 150), W_1 = 1, W_2 = 0, W_3 = 0 \), (d) Design a curve using five data points and three guide points plus three weight factors.
CONCLUSION

Designing curve depending on extra parameters in addition to the original data points will increase the complexity of the curve and give the designer higher flexibility to easily change the shape of the curve to the desired one without changing the original data points. In this paper, the area parametrization model on multiple cases, where a new models have been added to calculate the values of $t_i$ in the case of using only one guide at [1], the added models used the negative values of the guide to compute the parametric rather than using the only positive ones. In the other cases, the paper developed the area parametrization using two and three guide points with positive or negative values or both of them. The paper also used two and three weighting factors in addition to the guides also with the use of positive or negative values or both of them for the guides and weighting factors. When $\sum_{i=1}^{n} \hat{W}_i = 1$ the design is nicely nested more than the case of $\sum_{i=1}^{n} \hat{W}_i \neq 1$. The contribution of this paper allows the designer to easily generate curves complex design that not easy for counterfeiters to recreate without knowing the extra parameters that are secretly assigned by the designer, even in the case of knowing the original data points.

REFERENCES