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Numerical Solution for A Special Class of optimal Control Problem by using Hermite polynomial

Abstract- *In this paper, a numerical solution for solving a special class of optimal control problems is considered. The main idea of the solution is to parameterize the state space by approximating the state function using a linear combination of Hermite polynomial with unknown coefficients an iterative method is proposed in order to facilitate the computation of unknown coefficients. Some illustrated examples are included to test the efficiency of algorithm.*

Keywords- *Hermite polynomial, Optimal control problems, State parameterization.*

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1. Introduction

Optimal control has many applications in every area of science and engineering. And has been studied by many researches[1-4].

Since the analytic solution is not always available for optimal control problems, therefor a numerical solution must be found. Numerical methods for solving optimal control problem are vary in their approach and complexity. in [5] ,the authors suggested a new algorithm for solving optimal control problems and controlled duffing oscillator using Chebyshev polynomial as a basis function. While numerical solution for solving optimal control problems based on state parameterization technique were consider in [6] and [7]. Furthermore the fundamental of control parameterization method and solving its various applications were introduced in [8]. In addition, control parameterization technique for discrete value control problems was considered in [9].

In recent year different approximate methods and many algorithms has been introduced to solve the optimal control problems [10-13].

The organization of this paper is presented into the following sections. In section 2 the Hermite polynomial which are used as a basis function are reviewed briefly. Section 3, is about mathematical formulation of optimal control problem.in section4, the proposed algorithm is derived. While section 5 includes numerical example and results. Finally, the paper is concluded in section 6.

2. Hermite polynomials

In mathematics, the Hermite polynomials are a classical orthogonal polynomial sequence that arises in probability, such as the Edgeworth series, in combinatorics, as an example of an Appell sequence, obeying the umbral calculus, and in physics, where they give rise to the Eigenstates of the quantum harmonic oscillator. They are named in honor of Charles Hermite."

" in a sense to be described below, of the form

$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2}$ For n=1,2,3,…..

The first four Hermite polynomials are

$H_0(t) = 1$

$H_1(t) = 2t$

$H_2(t) = 4t^2 - 2$

$H_3(t) = 8t^3 - 12t$

$H_4(t) = 16t^4 - 48t^2 + 12$"

1-1 Definition: "For n ∈ N, we define Hermite polynomials $H_n(t)$ by

$$\sum_{n=0}^{\infty} \frac{H_n(t)}{n!} r^n = e^{2tr-r^2} \text{for } |r| < \infty \quad (1)$$

To find $H_n(t)$ expand the right hand side of (1) as a Maclaurin series in r and equate coefficients. From Equation (1) we derive the closed expression

$$H_n(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n!}{k!(n-2k)!} (2t)^{n-2k} \quad (2)$$

Where $\lfloor t \rfloor$ denoted the largest integer less than or equal to t .checking with n=0,1,2,…. We find that (2) yields the expected Hermite polynomials."

3. Mathematical formulate

The process illustrated by the following system of nonlinear differential equation on the final time interval [0,1] is consider

$$u(t) = f(t, x(t), \dot{x}(t)) \quad (3)$$

With initial condition $x(0)= x_0$,$x(1)=x_1$ (4)

Where $x(.): [0,1] \to R$ is the state variable, $u(.): [0,1] \to R$ is the control variable, and f is a real valued continuously differential function.

Along with the controlled process (3-4) a cost functional of the form

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$J = \int_0^1 L(t, x(t), u(t))dt$ (5)
is defined.

There are admissible control are always assume that pass through $(0,x_0)$and $(1,x_1)$ and in the set of controls, the control variable is searched which minimizes J and call it optimal control.

4. The proposed algorithm

The following approximate for x(.) Is first consider which is in terms the Hermite polynomials $H_k(t)$, $k = 0,1,2$

$x_1(t) = a_0H_0(t) + a_1H_1(t) + a_2H_2(t)$ (6)

Using the boundary condition (4), yields:

$a_0 = x_0 + 2a_2$ and $a_1 = \frac{x_1-x_0}{2} - 2a_2$ (7)

By substitution of (7) into (6), we obtain

$x_1(t) = a_2H_2(t) + (\frac{x_1-a_0}{2} - a_2)H_1(t) + (x_0 + 2a_2)H_0(t)$ (8)

The control variable u(t) are then obtained using eq.(3). Then, substituting $x_1(t)$ and u(t), we obtain J as a function of a_2. The solution of the optimal control problem (3-4) is $J(a^*)$ (a^* is the value which minimizes $J(a_2)$).

The state and control variables are also found from a^* approximately.

In the second step, the following approximated is use

$x_2(t) = x_1(t) + a_1H_1(t) + a_2H_2(t) + a_3H_3(t)$ (9)

Using the boundary conditions (4) one can obtain

$x_2(0) = x_1(0) + a_1H_1(0) + a_2H_2(0) + a_3H_3(0)$ (10)

$x_2(1) = x_1(1) + a_1H_1(1) + a_2H_2(1) + a_3H_3(1)$ (11)

From (10-11) we have

$a_2 = 0$ and $a_1 = 2a_3$ (12)

In this case the solution of optimal control problem (3-4)is $J(a^*)$ where a^* is the value which minimizes $J(a_3)$.

In general, the approximate solution in the n^th step will be

$x_n(t) = x_{n-1}(t) + a_{n-1}H_{n-1}(t) + a_nH_n(t) + a_{n+1}H_{n+1}(t)$ (13)

Using the first condition $x(0)=x_0$ to get

$x_n(0) = x_{n-1}(0) + a_{n-1}H_{n-1}(0) + a_nH_n(0) + a_{n+1}H_{n+1}(0)x_0 = x_0 + a_{n-1}H_{n-1}(0) + a_nH_n(0) + a_{n+1}H_{n+1}(0)a_{n-1}H_{n-1}(0) + a_nH_n(0) + a_{n+1}H_{n+1}(0) = 0$ (14)

Form the second condition of (4) we obtained

$a_{n-1}H_{n-1}(1) + a_nH_n(1) + a_{n+1}H_{n+1}(1) = 0$ (15)

We solve the equation (14) and (15) simultaneously to obtain a_{n-1} and a_n as a function of a_{n+1} as follows:

Multiply eq.(14) and (15)by $H_n(1)$and $H_n(0)$ respectively ,yields:

$H_n(1)(a_{n-1}H_{n-1}(0) + a_nH_n(0) + a_{n+1}H_{n+1}(0)) = 0$ $H_n(0)(a_{n-1}H_{n-1}(1) + a_nH_n(1) + a_{n+1}H_{n+1}(1)) = 0$From the above equations ,one can get

$a_{n-1} = \frac{H_n(0)H_{n+1}(1)-H_n(1)H_{n+1}(0)}{H_{n-1}(0)H_n(1)-H_{n-1}(1)H_n(0)}a_{n+1}$ (16)

$a_n = \frac{H_{n-1}(0)H_{n+1}(1)-H_{n+1}(1)H_{n+1}(0)}{H_{n-1}(1)H_n(0)-H_{n-1}(0)H_n(1)}a_{n+1}$ (17)

The denominator in Eq.(16) and (17) are not zero as illustrate in the following lemma.

<u>Lemma (1):</u>

The result of

$H_{n-1}(0)H_n(1) - H_{n-1}(1)H_n(0)$ (18)

is not zero.

Proof : If n is even ,then (18)becomes

$H_{2m-1}(0)H_{2m}(1) - H_{2m-1}(1)H_{2m}(0)$, m=0,1,2

Since we have $H_{2m}(0) = (-1)^m \frac{(2m)!}{m!}$

And $H_{2m-1}(0) = 0$, Therefor

$H_{2m-1}(1)H_{2m}(0) - H_{2m-1}(0)H_{2m}(1)$
$= -(-1)^m \frac{(2m)!}{m!}H_{2m}(1) = (-1)^{m+1} \frac{(2m)!}{m!}H_{2m}(1)$

Since $H_{2m}(1) \neq 0$

$H_{2m-1}(1)H_{2m}(0) - H_{2m-1}(0)H_{2m}(1) \neq 0$

Now if n is odd ,then $H_n(0) = 0$

$\Rightarrow H_{n-1}(1)H_n(0) - H_{n-1}(0)H_n(1) = -H_{n-1}(0)H_n(1)$ And

$H_{n-1}(0) = H_{2m}(0)$ $m = 0,1,2,\ldots$

Hence

$$-H_{2m}(0)H_n(1) = -(-1)^m \frac{(2m)!}{m!}H_n(1) = (-1)^{m+1}\frac{(2m)!}{m!}H_n(1)$$

Therefore

$H_{n-1}(1)H_n(0) - H_{n-1}(0)H_n(1) \neq 0$

The proposed algorithm can be summarized by the following steps:

Step 1: Choose an $\varepsilon > 0$.

Step 2: For n=1 , calculate :
$x_1(t) = a_2H_2(t) + (\frac{x_1-a_0}{2} - a_2)H_1(t) + (x_0 + 2a_2)H_0(t)$And then calculate a_2.

Step 3: For n=2 , calculate
$x_2(1) = x_1(1) + a_1H_1(1) + a_2H_2(1) + a_3H_3(1)$
Set $a_2=0$ and $a_1=2a_3$ calculate a_3.

Step 4: For $n \Rightarrow n + 1$, calculate

$$x_n(t) = x_{n-1}(t) + a_{n-1}H_{n-1}(t) + a_nH_n(t) + a_{n+1}H_{n+1}(t)$$

Set $a_{n-1} = \frac{H_n(0)H_{n+1}(1)-H_n(1)H_{n+1}(0)}{H_{n-1}(0)H_n(1)-H_{n-1}(1)H_n(0)}a_{n+1}$

$a_n = \frac{H_{n-1}(0)H_{n+1}(1)-H_{n+1}(1)H_{n+1}(0)}{H_{n-1}(1)H_n(0)-H_{n-1}(0)H_n(1)}a_{n+1}$Calculate a_{n+1} .

5. Numerical Examples

The efficiency of the proposed algorithms is the illustrated by same examples which have analytical solutions, so that the validation of the method can be allowed by comparing with the results of the exact solution.

Example (1)
This example concerns with the minimization of

$$J=\int_0^1 (x(t)-\frac{1}{2}u^2(t))dt \qquad (19)$$

Subject to

$$\dot{x}(t)=u(t)-x(t) \qquad (20)$$

With boundary conditions

$$x(0)=0, \qquad x(1)=\frac{1}{2}\left(1-\frac{1}{e}\right)^2 \qquad (21)$$

Where the analytical solution is:

$$x(t)=1-\frac{1}{2}e^{t-1}+\left(\frac{1}{3e}-1\right)e^{-t} \qquad (22)$$

$$u(t)=1-e^{t-1} \qquad (23)$$

Consider on approximation of $x_1(t)$ to be:

$$x_1(t)=a_0H_0(t)+a_1H_1(t)+a_2H_2(t) \qquad (24)$$

Using the boundary conditions (21) yields:

$$a_0=2a_2 \qquad (25)$$

$$a_1=\frac{1}{4}\left(1-\frac{1}{e}\right)^2-2a_2 \qquad (26)$$

Relations (25-26) are substituted into (24) to get the

$$x_1(t)=2a_2H_0+\left(\frac{1}{4}\left(1-\frac{1}{e}\right)^2-2a_2\right)H_1+a_2H_2 \qquad (27)$$

The control variable u(t) can be found from Eq.(20) with the use of Eq.(24) to be

$$u(t)=2a_2\dot{H}_0+\left(\frac{1}{4}\left(1-\frac{1}{e}\right)^2-2a_2\right)\dot{H}_1+a_2\dot{H}_2+2a_2H_0+\left(\frac{1}{4}\left(1-\frac{1}{e}\right)^2-2a_2\right)H_1+a_2H_2 \qquad (28)$$

Then substituted the Eqs.(24)and (26) into Eq.(19) ,we obtain J as a function of a2

$$J=\frac{1038}{1947}-\frac{44}{15}a_2^2-\frac{1081}{1801}a_2$$

The value which minimize J is $a^*=a_2=$ -0.1023 then $J(a^*)=0.08401526$
In addition a_0 = -0.2046 and a_1= 0.3045.
The state and control variables can be calculated approximately as

$$x_1=\frac{5485}{9007}t-\frac{7370}{1801}t^2$$

$$u=\frac{5485}{9007}-\frac{9428}{4504}t-\frac{7370}{1801}t^2$$

Now the approximated solution can be modified as below

$$x_2(1)=x_1(1)+a_1H_1(1)+a_2H_2(1)+a_3H_3(1) \qquad (29)$$

And the results of repeated the above procedure are summarization as follows :

$$a_1=-\frac{3599}{1801}+\frac{1}{4}\left(1-\frac{1}{e}\right)^2+2a_3$$

$$x_2(t)=\frac{5485}{9007}t-\frac{7370}{1801}t^2+\left(-\frac{3599}{1801}+\frac{1}{4}\left(1-\frac{1}{e}\right)^2+2a_3\right)H_1+a_2H_2+a_3H_3 \quad a_3=2.3752e-04$$

$$a_1=4.7504e-04$$

$$x_2=\frac{1399}{2306}t-\frac{7370}{1801}t^2+\frac{2191}{1153}t^3$$

$$u=\frac{1399}{2306}-\frac{4871}{2306}t-\frac{4651}{1153}t^2+\frac{2191}{1153}t^3$$

And the value of J*: 0.08401684.
The approximate results are listed in table (1) and are plotted in Figure (1) and Figure (2).

Table (1)

time	**Hermite polynomial**	
t	x	u
0	0	0.6089
0.1	0.0568	0.5839
0.2	0.1054	0.5507
0.3	0.1459	0.5093
0.4	0.1781	0.4597
0.5	0.2022	0.4020
0.6	0.2181	0.3360
0.7	0.2258	0.2619
0.8	0.2253	0.1796
0.9	0.2166	0.0891
1	0.1998	-0.0096
J^*	0.08401684	
exact	0.0840456	

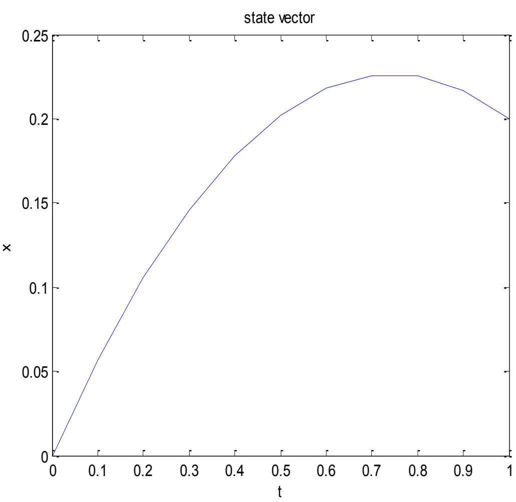

Figure (1) State vector

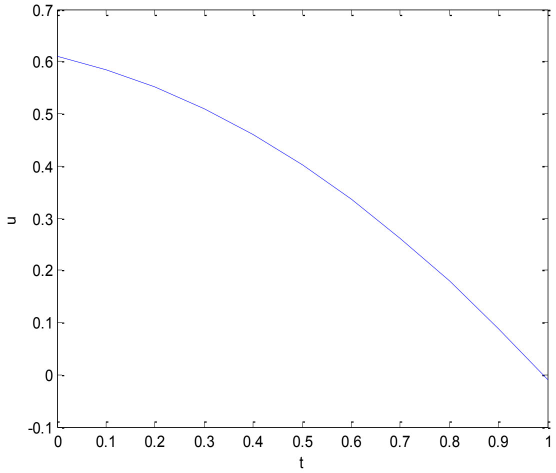

Figure (2) Optimal control vector

Example (2): The performance index to be minimized is

$$J = \frac{1}{2}\int_0^1 (3x^2(t) + u^2(t))dt \quad (30)$$

$$\dot{x}(t) = u(t) - x(t) \quad (31)$$

$$x(0) = 0 \qquad x(1) = 2 \quad (32)$$

we will solved by expanding *x(t)* into two order Hermite series.

N =2 the state variable can be written as

$$x_1(t) = a_0H_0(t) + a_1H_1(t) + a_2H_2(t) \quad (33)$$

And the same steps above in example (1) we obtain

$$x_1 = \frac{4}{7}t + \frac{10}{7}t^2$$

$$u = \frac{4}{7} + \frac{24}{7}t + \frac{10}{7}t^2$$

and the result value of J is 6.1905.

The modified equation of x1 is

$$x_2 = x_1 + a_1H_1 + a_2H_2 + a_3H_3 \quad (34)$$

And re-sequencing solution steps, such as the first example. The value of J is 6.0693

The approximate results are listed in table (2)

Table (2)

n	J
1	6.6667
2	6.1905
2(modified method)	6.0693

Example (3):

Consider the following quadratic optimal control problem

Minimize

$$J = \int_0^1 (x^2(t) + u^2(t))dt \quad (35)$$

$$\dot{x}(t) = u(t) \quad (36)$$

$$x(0) = 0 \qquad x(1) = \frac{1}{2} \quad (37)$$

we approximate the state variable by 2nd order series of unknown parameters.

$$x_1 = a_0H_0 + a_1H_1 + a_2H_2 \quad (38)$$

The first result of x1 is

$$x1 = \frac{17}{44}t + \frac{5}{44}t^2 \quad \text{and}$$

$$u = \frac{17}{44} + \frac{10}{44}t$$

the value of J =0.3286 .And the value of J becomes 0.32857867 after use the modified equation $x_2 = x_1 + a_1H_1 + a_2H_2 + a_3H_3$

And we can use other the approximated solution x as

$$x_2 = x_1 + a_0H_0 + a_2H_2 + a_3H_3$$

And the value of J=0.328587046. See Table (3).

Table (3)

n	J
2	0.3286
2(modified method)	0.32857867

6. Conclusion

The proposed algorithm for treating optimal control problem depending on Hermite polynomial and their propertied provided a simple way to obtain an optimal control with fast convergence.

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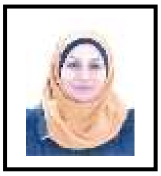

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