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1. Introduction and Preliminaries

Let $B$ be a nonempty subset of a Banach space $M$. A map $T$ on $B$ is called nonexpansive map if $\|T a - T b\| \leq \|a - b\|$ for all $a, b \in B$. It is called quasi-nonexpansive map [1] if $\|T a - b\| \leq \|a - p\|$ for all $a \in B$ and for all $p \in F(T)$, denote by $F(T)$ the set of all fixed point of $T$.

In 2008, a new condition for maps, called condition (C) was introduced by Suzuki [2], which is stronger than quasi-nonexpansive and weaker than nonexpansive, and given some results about fixed point for map satisfying condition (C). Dhompongsa et al [3] and Phuengrattana [4] studied fixed point theorems for a map satisfying condition (C). Weak convergence theorem for a map satisfying condition (C) in uniformly convex Banach space are proved by Kuhn and Suzuki [5]. Recently, Garcia-Falset et al [6] introduced two new generalization of condition (C), called condition $(E_2)$, condition $(C_2)$ and studied the existence of fixed points and also their asymptotic behavior. For approximating common fixed point of two maps, Takahashi and Tamura [7] studied the following Ishikawa iteration scheme for two nonexpansive maps.

\[
\begin{align*}
 a_{n+1} &= (1 - \alpha_n) a_n + \alpha_n T b_n \\
 b_{n+1} &= (1 - \beta_n) a_n + \beta_n T a_n \\
 a_n &\in B \\
 b_n &\in B
\end{align*}
\]

for all $n \in N, (\alpha_n)$ and $(\beta_n) \in [0,1]$.

The aim of this paper is to study weak convergence of the Picard-Mann iteration scheme, Liu et al iteration scheme for approximating common fixed point of generalized nonexpansive and quasi-nonexpansive maps and give some corollaries.

Abstract - In this paper, we established weak convergence theorems by using appropriate conditions for approximating common fixed points and equivalence between the convergence of the Picard-Mann iteration scheme and Liu et al iteration scheme in Banach spaces. As well as, numerical examples are given to show that Picard-Mann is faster than Liu et al iteration schemes.

Keywords - Banach space, weak convergence, common fixed points.

Weak Convergence of Two Iteration Schemes in Banach Spaces

Definition (1.1): A Banach space $M$ is called satisfying:

1-Opial’s condition [10] if for any sequence $(a_n)$ in $M$, is weakly convergent to $a$ implies that
\[
\lim_{n \to \infty} \|a_n - a\| < \lim_{n \to \infty} \|a_n - b\|
\]
for all $b \in M$ with $a \neq b$.

2-Kadec-Klee property [11] if for every sequence $(a_n)$ in $M$ converging weakly to $(a)$ together with $\|a_n\|$ converging strongly to $\|a\|$ imply that $(a_n)$ converges strongly to a point $a \in M$.

Definition (1.2) [12]: A map $T : B \to M$ is said to be generalized nonexpansive if there are
theorem (1.9)[15]: let M be a uniformly convex Banach space. Then for any r and e with r ≥ e ≥ 0 and elements a, b ∈ M such that \(\|a\| ≤ r, \|b\| ≤ r, \|a - b\| ≥ e, \exists \delta = \delta(\frac{e}{r}) > 0\) such that
\[
\frac{\|a + b\|}{2} ≤ r \left(1 - \delta \left(\frac{e}{r}\right)\right).
\]

Proposition (1.10)[16]: let B be a closed convex set in a Banach space M. If \((a_n)\) converges weakly to a for some sequence \((a_n)\) in M, then a ∈ M.

Lemma (1.11)[17]: let \((\mu)_{n=0}^\infty\) and \((\rho)_{n=0}^\infty\) be nonnegative real sequences satisfying the inequality:
\[
\mu_{n+1} ≤ (1 - \sigma_n)\mu_n + \rho_n
\]
where \(\sigma_n ∈ (0,1), \forall n ≥ n_0, \sum_{n=1}^\infty \sigma_n = \infty\) and \(\rho_n \to 0\) as \(n \to \infty\). Then \(\lim_{n \to \infty} \mu_n = 0\).

Lemma (1.12)[13]: Let M be a uniformly convex Banach space and \(0 ≤ L ≤ r ≤ K < 1, \forall n ∈ N\). Suppose that \((a_n)\) and \((b_n)\) are two sequences of M such that:
\[
\lim_{n \to \infty} \|a_n\| ≤ m, \lim_{n \to \infty} \|b_n\| ≤ m
\]
and
\[
\|b_n\| \to \infty, \|b_n\| \to m \quad \text{hold for some} \ m ≥ 0. \text{Then} \lim_{n \to \infty} \|a_n - b_n\| = 0.
\]

Lemma (1.13)[18]: Let B be a nonempty convex subset of a uniformly convex Banach space. Then there is a strictly increasing continuous function \(f : [0, \infty) \to [0, \infty)\) with \(f(0) = 0\) such that for each lipschitzian map \(T : B \to B\) with lipschitz constant K:
\[
\|tTx + (1-t)Ty - T(tx + (1-t)y)\| ≤ Kf^{-1} \left(\frac{\|x - y\|}{1 - K\|Tx - Ty\|}\right),
\]
\(\forall x, y ∈ B\) and \(t ∈ [0,1]\).

Lemma (1.14)[18]: Let M be a uniformly convex Banach space such that its dual \(M^*\) satisfies the Kadec-Klee property. Assume that \((a_n)\) bounded sequence in M such that \(\lim_{n \to \infty} \|ta_n + (1-t)p_1 - p_2\| = 0\) exists \(\forall t ∈ [0,1]\) and \(p_1, p_2 ∈ \text{W}(a_n)\), then \(p_1 = p_2\).

2. The Main Results
Proposition (2.1): Let $B$ be a closed convex bounded of uniformly convex Banach space, $T:B \to M$ is a generalized nonexpansive map and $a_0,a_1 \in B$, $a_0 \neq a_1 \forall t \in [0,1], a_t = ta_0 + (1-t)a_1$. If $t > 0, \exists a(\epsilon) > 0$ such that $\|Ta_0 - a_1\| \leq \epsilon$ and $\|Ta_t - a_t\| \leq \epsilon$. Then $\|Ta_t - a_t\| \leq a(\epsilon)$ and $a(\epsilon) \to 0$ as $\epsilon \to 0$.

Proof: Assume that (3) holds with $a_0 \neq a_1$ and $0 < t < 1$. Let $i = 0, 1$ such that $\|a_i - (a_t + Ta_t)/2\| \geq \|a_i - a_t\|$. If not, would have the contradiction $\|a_i - a_0\| \leq \sum_{i=0}^{1} \|a_i - a_t + Ta_t/2\|$.

Since $a_1 \neq a_0$ we have $r = \|a_t - a_1\| > 0, n = \|a_t - Ta_t\|, m = \|a_t - a_0\|$. Since $T$ is generalized nonexpansive mapping $\|Ta_t - a_t\| \leq \|Ta_t - a_t\| + \|Ta_t - a_t\| \leq \delta a_t + a_t + \mu(\|a_t - Ta_t\|) + \omega(\|a_t - Ta_t\|) + \|Ta_t - a_t\| \leq \delta r + \mu(a(\epsilon) + \epsilon) + n^2 + m + \epsilon$.

Let $\epsilon_t$ denote the strictly monotone increasing function to $\epsilon(t)$. The diameter of $M$ denotes by $diam(M)$, by theorem (2.10), we have $\|Ta_t - a_t\| \leq sup_{t \in [0,1]}(w + \epsilon)(\epsilon/\epsilon + \epsilon)$.

The $a(\epsilon)$ defined here has desired properties. First $a(\epsilon) \geq \epsilon_t(1) = 2\epsilon_t$ for $w = 0$. Forming the supremum separately over the two intervals $[0,\sqrt{\epsilon} - \epsilon]$ and monotonicity of $\epsilon(\cdot)$, that $a(\epsilon) \leq \max\{\sqrt{\epsilon_t}(1), d(M) + \epsilon\} \to 0$ as $\epsilon \to 0$.

Since $a(\epsilon) > 2\epsilon_t$ for $\|Ta_t - a_t\| \leq a(\epsilon)$ as $a(\epsilon) \to 0$ as $\epsilon \to 0$.

Hence (3) holds for the remaining cases $a_1 \neq a_0, t = 0, 1$ and $a_0 = a_1$.

Theorem (2.2): Let $B$ be a closed, bounded and convex subset of uniformly convex Banach space $M$, then the operator $I - T$ is demiclosed on $B$.

Proof: We show that for any sequence $(a_n)$ in $M$, if $(a_n)$ converges weakly to $a$ and $(I - T)(a_n)$ converges strongly to $0$ as $n \to \infty$, then $a \in M$ and $(I - T)(a) = 0$.

By proposition (1.10), we get $a \in M$. For $\epsilon_0 \in (0,1)$ choose a sequence $(\epsilon_n)$ such that $\epsilon_n \leq \epsilon_{n-1}$ and $a(\epsilon_n) \leq \epsilon_{n-1}, \forall n \in N$.

This is possible because $a(\epsilon) \to 0$ as $\epsilon \to 0$.

Choosing a subsequence of $(a_n)$ if necessary, we have $\|Ta_n - a_n\| \leq \epsilon_n, \forall n \in N$.

Then $\|Ta_n - a_n\| \leq \epsilon_0, \forall b \in conv(a_n, \epsilon_0)$.

Now let $b_1 \in conv(a_m, \epsilon_n) \forall 1 \leq m < n$, by hypothesis $\|Ta_m - a_m\| \leq \epsilon_m$ and $\|Ta_n - a_n\| \leq \epsilon_n, \epsilon_n \leq \epsilon_n$.

Then $\|Ta_1 - a_1\| \leq a(\epsilon_m) \leq \epsilon_m$.

ii) Let $b_2 \in conv(a_k, \epsilon_n) \forall 1 \leq k < n$. The key is that $b_2 \in conv(a_k, b_1)$ since $b_1 \in conv(a_m, a_n)$. By (i) $\|Ta_1 - b_1\| \leq \epsilon_{m-1}, \epsilon_{m-1} \leq \epsilon_{k}, \epsilon_{k}$.

$\|Ta_k - a_k\| \leq \epsilon_k$ and $\|Ta_1 - b_1\| \leq \epsilon_k$.

Hence $\|Ta_1 - a_1\| \leq \epsilon_0$ since $\epsilon_0$ can be any arbitrary small. $Ta_1 - a_1 = 0$.

Not only is the map $a \to Ta$ generalized nonexpansive, but for fixed point $a \in conv(a_1, a_n)$.

This implies that $I - T$ is demiclosed.

Lemma (2.3): Let $T:B \to B$ be a quasi-nonexpansive map and $S:B \to B$ be Lipschitzian and generalized nonexpansive maps. Let $(a_n)$ be as in (1) where $(a_n) \in (0,1)$. $(e_n)$ be as in (2) where $(a_n)$ and $(b_n) \in [0,1]$.

If $F(T,S) \neq \emptyset$, then $\lim_{n \to \infty} \|a_n - a^*\|$ and $\lim_{n \to \infty} \|z_n - a^*\|$ both exist for all $a^* \in F(T,S)$.

Proof: Let $a^* \in F$.

i) $\|a_{n+1} - a^*\| = \|Sb_n - a^*\| \\
\leq \delta \|b_n - a^*\| + \mu\|b_n - Sb_n\| + \omega\|b_n - a^*\| + \omega\|Sb_n - a^*\| \\
\leq (\delta + \mu + \omega)\|b_n - a^*\| + (\mu + \omega)\|Sb_n - a^*\| \\
\leq (\delta + 2\mu + 2\omega)\|b_n - a^*\| + \omega\|Sb_n - a^*\| \\
\leq \|b_n - a^*\| \\
\leq (1 - a_n)^{\|a_n - a^*\|} + a_n\|a_n - a^*\| \\
= \|a_n - a^*\|$. 

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then \( \lim_{n \to \infty} \|a_n - a^*\| \) exists \( \forall a^* \in F(T, S) \).

\[
\|u_n - a^*\| \leq (1 - \beta_n)\|Sz_n - a^*\| + \beta_n\|Tz_n - a^*\|
\]

this implies to
\[
\lim \sup_{n \to \infty}\|b_n - a^*\| \leq c
\]

\[
\text{moreover } d = \lim_{n \to \infty}\|a_{n+1} - a^*\| \leq \|b_n - a^*\|
\]

then
\[
c \leq \inf_{n \to \infty}\|b_n - a^*\|
\]

By (5) and (6), we get
\[
\lim_{n \to \infty}\|b_n - a^*\| = c
\]

Next consider
\[
c = \|b_n - a^*\|
\]

By applying lemma (1.12), we get
\[
\lim_{n \to \infty}\|a_n - Ta_n\| = 0
\]

\[
c = \lim_{n \to \infty}\|a_{n+1} - a^*\| = \lim_{n \to \infty}\|Sb_n - a^*\|
\]

By applying lemma (1.12), we get
\[
\lim_{n \to \infty}\|Sa_n - STa_n\| = 0
\]

Now
\[
\|Sa_n - a_n\| \leq \|Sa_n - STa_n\| + \|STa_n - a_n\|
\]

By using the hypothesis condition, we have
\[
\|Sa_n - a_n\| \leq 2\|Sa_n - STa_n\| \to 0 \text{ as } n \to \infty.
\]

Thus
\[
\lim_{n \to \infty}\|Sa_n - a_n\| = 0.
\]

**Lemma (2.4):** Let \( T: B \to B \) be a quasi-nonexpansive map and \( S: B \to B \) be Lipschitzian, generalized nonexpansive maps and affine and \( (a_n) \) be as in (1). Suppose that the following condition \( \|a - Tb\| \leq \|S(a - Tb)\| \), \( \forall a, b \in B \) holds. If \( F(T, S) \neq \emptyset \), then
\[
\lim_{n \to \infty}\|Ta_n - a_n\| = \lim_{n \to \infty}\|Sa_n - a_n\| = 0
\]

**Proof:** Let \( a^* \in F(T, S) \).

By lemma (2.3.i), \( \lim_{n \to \infty}\|a_n - a^*\| \) exists. Suppose that \( \lim_{n \to \infty}\|a_n - a^*\| = c, \forall c \geq 0 \).

If \( c = 0 \), there is nothing to prove.

Now suppose \( c > 0 \),
\[
\|a_{n+1} - a^*\| = \|Sa_n - a^*\| \leq \|b_n - a^*\|
\]

By lemma (2.3.i), we show that \( \|b_n - a^*\| \leq \|a_n - a^*\| \)

**Lemma (2.5):** Let \( T: B \to B \) be a quasi-nonexpansive map, \( S: B \to B \) be Lipschitzian and generalized nonexpansive maps and \( (z_n) \) be as in (2). Suppose that the following condition \( \|a - Tb\| \leq \|S(a - Tb)\| \), \( \forall a, b \in B \) holds. If \( F(T, S) \neq \emptyset \), then
\[
\lim_{n \to \infty}\|Tz_n - z_n\| = \lim_{n \to \infty}\|Sz_n - z_n\| = 0.
\]

**Proof:** Let \( a^* \in F(T, S) \).

By lemma (2.3.i), \( \lim_{n \to \infty}\|z_n - a^*\| \) exists. Suppose that \( \lim_{n \to \infty}\|z_n - a^*\| = c, \forall c \geq 0 \).

If \( c = 0 \), there is nothing to prove.

Now suppose \( c > 0 \),
\[
\|z_{n+1} - a^*\| = \|Sa_n - a^*\| \leq \|b_n - a^*\|
\]

By lemma (2.3.i), we show that \( \|b_n - a^*\| \leq \|a_n - a^*\| \)

this implies to
\[
\|z_{n+1} - a^*\| \leq (1 - \alpha_n)\|Sz_n - a^*\| + \alpha_n\|Tz_n - a^*\|
\]

By applying lemma (1.12), we get
\[
\lim_{n \to \infty}\|Sz_n - a^*\| = 0
\]

\[
\|a_{n+1} - a^*\| = \|(1 - \alpha_n)Sz_n + \alpha_nTu_n - a^*\| \leq \|Sz_n - a^*\| + \alpha_n\|Sz_n - Tu_n\|
\]

\[
\text{this implies to}
\]
\[
c \leq 0
\]

\[
\text{and} \|S_z - a^*\| \leq \|z_n - a^*\|
\]

therefore
\[
\lim \sup_{n \to \infty}\|S\| = c
\]
By (7) and (8), we have
\[ \lim_{n \to \infty} \|Sz_n - a^*\| = c \]
\[ \|Sz_n - a^*\| \leq \|Sz_n - Tu_n\| + \|Tu_n - a^*\| \]
that yields to
\[ c \leq \liminf_{n \to \infty} \|u_n - a^*\| \tag{9} \]
and
\[ \|u_n - a^*\| \leq (1 - \beta_n)\|Sz_n - a^*\| + \beta_n\|Tz_n - a^*\| \]
\[ = \|z_n - a^*\| \]
Now
\[ \lim_{n \to \infty} \sup_{m \to \infty} \|u_n - a^*\| \leq c \] \tag{10}
By (9) and (10), we have
\[ \lim_{n \to \infty} \|u_n - a^*\| = c \]
\[ \|u_n - a^*\| \leq (1 - \beta_n)\|Sz_n - a^*\| + \beta_n\|Tz_n - a^*\| \]
By applying lemma (1.12), we obtain
\[ \lim_{n \to \infty} \|Sz_n - Tz_n\| = 0 \]
Now
\[ \|Sz_n - z_n\| \leq \|Sz_n - Tz_n\| + \|Tz_n - z_n\| \]
By using the hypothesis condition, we get
\[ \|Sz_n - z_n\| \leq 2\|Sz_n - Tz_n\| \to 0 \text{ as } n \to \infty \]
and
\[ \|Tz_n - z_n\| \leq \|Tz_n - Sz_n\| + \|Sz_n - z_n\| \]
\[ \leq 2\|Tz_n - Sz_n\| \to 0 \text{ as } n \to \infty \].
Hence
\[ \lim_{n \to \infty} \|Tz_n - z_n\| = \lim_{n \to \infty} \|Sz_n - z_n\| = 0. \]

**Lemma (2.6):** Let \( T; B \to B \) be Lipschitzain and quasi-nonexpansive maps and \( S; B \to B \) be lipschitzain and generalized nonexpansive maps. Then for \( a_1^*, a_2^* \in F(T, S), (a_n) \) be as in (1) and \((z_n)\) be as in (2) such that
\[ \lim_{n \to \infty} \|Ta_n + (1 - t)a_1^* - a_2^*\| \text{ and } \lim_{n \to \infty} \|Tz_n + (1 - t)a_1^* - a_2^*\| = 0, \forall t \in [0, 1]. \]

**Proof:** Now to prove \( \lim_{n \to \infty} \|Ta_n + (1 - t)a_1^* - a_2^*\| \) exists and equal to zero, by lemma (2.3.i) \( \lim_{n \to \infty} \|a_n - a^*\| \) exists, \( \forall a^* \in F(T, S) \) and \((a_n)\) is bounded.

Then there is a real number \( L > 0 \) such that \( (a_n) \subseteq D = B_{L}(0) \cap B \), so that \( D \neq \emptyset \) is a closed convex bounded subset of \( B \).

Put \( y_n(t) = \|Ta_n + (1 - t)a_1^* - a_2^*\| \).
Notice that \( y_n(0) = \|a_1^* - a_2^*\| \) and \( y_n(1) = \|a_n - a_2^*\| \) exists by lemma (2.3.i).

**Defin R_n; D \to D, \forall n \in N, R_n a = S(1 - \alpha_n)a_n + \alpha_n Ta_n \forall a \in D.**
\[ \|R_n a - R_n b\| = \left\| S(1 - \alpha_n)a_n + \alpha_n Ta_n \right\| \]
\[ = \left\| -S(1 - \alpha_n)b_n + \alpha_n Tb_n \right\| \]
\[ \leq (1 - \alpha_n)\|a_n - b_n\| + \alpha_n\|Ta_n - Tb_n\| \]
\[ \leq (1 - \alpha_n)\|a_n - a^*\| + \alpha_n\|Ta_n - a^*\| \]
\[ \leq (1 - \alpha_n)\|a_n - a^*\| + \alpha_n(1 - \alpha_n)\|b_n - a^*\| \]
\[ + \alpha_n\|a_n - a^*\| + \alpha_n\|b_n - a^*\| \]
\[ = \|a_n - a^*\| + \|b_n - a^*\| \]

Set \( W_{n,m} = R_{n+m}R_{n+m-1}...R_1 \) and
\[ b_{n,m} = W_{n,m}(ta_n + (1 - t)a_1^*) - W_{n,m}a_n + (1 - t)a_1^* \], \( \forall n, m \in N. \)

Then
\[ \|W_{n,m}a - W_{n,m}b\| \]
\[ \leq \|W_{n,m}a - a^*\| + \|W_{n,m}b - a^*\| \]
\[ \leq \|a - a^*\| + \|b - a^*\| \]
and
\[ \|W_{n,m}a - a^*\| \leq \|a - a^*\|, W_{n,m}a_n = a_n \text{ and } W_{n,m}a^* = a^*, \forall a^* \in F. \]
By lemma (1.13) there is a strictly increasing function continuous function \( f: [0, \infty) \to [0, \infty) \) with \( f(0) = 0 \) such that
\[ b_{n,m} \leq Kf^{-1}(\|a_n - a_1^*\| - \|W_{n,m}a_n\|) \]
\[ \leq Kf^{-1}(\|a_n - a_1^*\| - \|a_{n+m} - a_1^*\|) \]
since \( \lim_{n \to \infty} \|a_n - a^*\| \) exists \( \forall a^* \in F. \)
\[ \lim_{n \to \infty} \sup_{m \to \infty} b_{n,m} = 0 \text{ yields } \lim_{n \to \infty} \sup_{m \to \infty} b_{n,m} = 0. \]
Now,
\[ y_{n,m}(t) = \|ta_{n+m} + (1 - t)a_1^* - a_2^*\| \]
\[ = \|tW_{n,m}a_n + (1 - t)a_1^* - a_2^*\| \]
\[ = \left\| tW_{n,m}a_n + (1 - t)a_1^* - a_2^* + W_{n,m}(ta_n + (1 - t)a_1^*) - a_2^* \right\| \]
\[ -a_2^* + a_1^* - a_2^* \]
\[ \leq b_{n,m} + \|W_{n,m}(ta_n + (1 - t)a_1^*) - a_2^*\| \]
\[ \leq b_{n,m} + \|W_{n,m}(ta_n + (1 - t)a_1^*) - W_{n,m}a_2^*\| \]
\[ \leq b_{n,m} + y_n(t) \]
Now
\[ \lim_{n \to \infty} \sup_{m \to \infty} y_{n,m}(t) \leq \lim_{n \to \infty} \sup_{m \to \infty} b_{n,m} + y_n(t) \]
then
\[ \lim_{n \to \infty} \sup_{m \to \infty} y_{n,m}(t) \leq \lim_{n \to \infty} \inf y_n(t) \]
which implies that \( \lim_{n \to \infty} \|Ta_n + (1 - t)a_1^* - a_2^*\| \) exists \( \forall t \in [0, 1]. \)

Now to prove \( \lim_{n \to \infty} \|Tz_n + (1 - t)a_1^* - a_2^*\| \) exists. By lemma (2.3.i) \( \lim_{n \to \infty} \|z_n - a^*\| \) exists, \( \forall a^* \in F(T, S) \) and \((z_n)\) is bounded.

Then there is a real number \( L > 0 \) such that \( (z_n) \subseteq D = B_{L}(0) \cap B \) so that \( D \neq \emptyset \) is a closed convex bounded subset of \( B \).
Put $\gamma_n(t) = \|tz_n + (1 - t)a_n^* - a_n^*\|$
notice that $\gamma_n(0) = \|a_n^* - a_n^*\| and \gamma_n(1) = \|a_n - a_n^*\|$ exists by lemma (2.3.b).

Define $R_n:D \to D, \forall n \in N, R_nz = (1 - \alpha_n)Sz_n + \alpha_nT((1 - \beta_n)Sz_n + \beta_nTz_n)$
which is a mapping such that $\lim_{n \to \infty}z_n = 0$.

By lemma (2.4) we can prove that $\lim_{n \to \infty}z_n = 0$ respectively. By lemma (2.4)
unique weak subsequential limit in $F$.

Since $S$ and $T$.

both converge weakly to a common fixed point o

Theorem (2.7): Let $M$ be a uniformly convex Banach space satisfying Opial’s condition and
$T:B \to B$ be quasi-nonexpansive map with $(1 - T)$ demiclosed at zero, $S:B \to B$ be Lipschitz and
generalized nonexpansive maps and $(a_n), (z_n)$ as in lemma (2.4) and lemma (2.5), respectively. If $F(T, S) \neq \emptyset$, then $(a_n) and (z_n)$ converge weakly to a common fixed point of $S$ and $T$.

Proof: Since $(a_n)$ and $(z_n)$ are bounded and $M$ is reflexive. Then, there is a subsequence
of $(a_n)$ that converges weakly to a point $a^* \in B$. By lemma (2.4)
$\lim_{n \to \infty}S\alpha_n - \alpha_n = 0 = \lim_{n \to \infty}Ta_n - \alpha_n$ thus $a^* \in F(T, S)$.

To prove $(a_n)$ converges weakly to a point $a^*$. Assume that $(a_{nk})$ is another subsequence of $(a_n)$
that converges weakly to a point $b^* \in B$.

Then by lemma (2.6) $\lim_{n \to \infty}\|\alpha_n + (1 - t)a^* - b^*\| exists \forall t \in [0, 1]$.

By lemma (1.15) $a^* = b^*$. Then $(a_n)$ converges weakly to the point $a^* \in F(T, S)$.

Utilizing the same above argument to prove that $(z_n)$ converges weakly to the point $a^* \in F(T, S)$.

The following corollary as a special case of
\textbf{quasi-nonexpansive mapping is now obvious.}

\textbf{Corollary (2.8):} Let $M$ be a uniformly convex Banach space satisfying Opial’s condition and
$T:B \to B$ be satisfying condition $(C_2), S:B \to B$ be Lipschitz and
generalized nonexpansive maps and $(a_n), (z_n)$ as in lemma (2.4) and lemma (2.5), respectively. If $F(T, S) \neq \emptyset$, then $(a_n) and (z_n)$ converges weakly to a common fixed point of $S$ and $T$.

\textbf{Corollary (2.9):} Let $M$ be a uniformly convex Banach space and its dual $M^*$ satisfies the Kadec-Klee property and $T:B \to B$ be lipschitzain map and satisfying condition $(C_3)$ and $S:B \to B$ be lipschitzain and generalized nonexpansive maps and $(a_n), (z_n)$ be as in lemma (2.6). If $F(T, S) \neq \emptyset$, then $(a_n) and (z_n)$ converges weakly to a common fixed point of $S$ and $T$.

\textbf{Corollary (2.10):} Let $M$ be a uniformly convex Banach space satisfying Opial’s condition and
$T:B \to B$ be satisfying condition $(E_2), S:B \to B$ be generalized nonexpansive map and $(a_n), (z_n)$ be as in lemma (2.4) and lemma (2.5), respectively. If $F(T, S) \neq \emptyset$, then $(a_n) and (z_n)$...
converges weakly to a common fixed point of S and T.

Corollary (2.11): Let M be a uniformly convex Banach space and its dual $M^*$ satisfies the Kadec-Klee property and $T: B \to B$ be Lipschitzian map and satisfying condition (E$_2$) and $S: B \to B$ be Lipschitzian and generalized nonexpansive maps and \((a_n),(z_n)\) be as in lemma (2.6). If $F(T,S) \neq \emptyset$, then \((a_n)\) and \((z_n)\) converges weakly to a common fixed point of S and T.

3. Equivalence of Iterations

**Theorem (3.1):** Let $B$ be a nonempty closed convex subset of a Banach space $M$. Let $T: B \to B$ be a quasi-nonexpansive map, $S: B \to B$ be Lipschitzian and generalized nonexpansive maps and $a^* \in B$ be a common fixed point of $S$ and $T$. Let $(a_n)$ and $(z_n)$ be the Picard-Mann and Liu et al. iteration schemes defined in (1) and (2), respectively. Suppose $(a_n)$ and $(\beta_n)$ satisfied the following conditions:

1. $(a_n)$ and $(\beta_n) \in (0,1)$, $\forall n \geq 0$.
2. $\sum a_n = \infty$.
3. $\sum a_n \beta_n < \infty$.

If $z_0 = a_0$ and $R(T), R(S)$ are bounded, then the Picard-Mann iterative sequence $(a_n)$ converges strongly to $a^*$ $(a_n \to a^*)$ and the Liu et al. iterative sequence $(z_n)$ converges strongly to $a^*$ $(z_n \to a^*)$.

**Proof:** Since the range of $T$ and $S$ are bounded, let $M = \sup_{a \in B} \{||Ta||\} + ||a_0|| < \infty$

then

\[ ||a_n - z_n|| \leq M, ||b_n - z_n|| \leq M, ||u_n|| \leq M \]

therefore

\[ ||T a_n|| \leq M, ||T z_n|| \leq M \]

\[ ||a_{n+1} - z_{n+1}|| = ||Sb_n - (1 - a_n)Sz_n - a_n Tu_n|| \leq ||Sb_n - Sz_n|| + a_n ||Sz_n - Tu_n|| \leq ||Sb_n - a^*|| + ||Sz_n - a^*|| + a_n ||Sz_n - a^*|| + a_n ||Tu_n - a^*|| \]

\[ \leq \alpha ||b_n - a^*|| + \mu \{ ||b_n - Sb_n|| \} + \delta ||z_n - a^*|| \]

\[ + \omega \{ ||b_n - Sb_n|| \} + \delta ||z_n - a^*|| \]

\[ + \alpha_n ||z_n - a^*|| + a_n ||u_n - a^*|| \leq \delta ||b_n - a^*|| + \mu \{ ||b_n - Sb_n|| \} \]

\[ + \omega \{ ||b_n - Sb_n|| \} + \alpha_n ||z_n - a^*|| + a_n ||u_n - a^*|| \]

\[ \leq \delta ||b_n - a^*|| + \delta ||z_n - a^*|| + \mu \{ ||b_n - Sb_n|| \} \]

<table>
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Example (4.2): Let \( B = [-180,180] \), \( T,S:B \to B \) be a map defined by \( Ta = acos a \) and \( Sa = a \) \( \forall a \in B \). Choose \( \alpha_n = \frac{1}{3}, \beta_n = \frac{1}{5} \) \( \forall n \) with initial value \( a_1 = 30 \). The two iteration scheme converge to the same fixed point \( a^* = 0 \). It’s clear from table 2, that Picard-Mann converges faster than Liu et al.

Table 2: Numerical results corresponding to \( a_1 = 30 \) for 30 steps

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<tr>
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Finally, it is appropriate to ask a question about the possibility of employing the above results in finding solutions to problems such in [19] and [20]

References


