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Generalized Dependent Elements of Generalized Reverse Derivation on Semiprime Rings

Abstract- Let R be an associative ring, and $\check{S}:R\longrightarrow R$ be a map, if there exists an element $e\in R$ such that $\check{S}(u)e=[u,e]e$, for every $u\in R$, in this case e is called Generalized Dependent Element of $\check{S}$, and $\check{G}\text{-}D(\check{S})$ denote the set of all Generalized Dependent Elements of $\check{S}$. In this paper the result proved, let R be semiprime ring, and $F : R\longrightarrow R$ is a generalized reverse derivation, related with derivation d, then $e\in\check{G}\text{-}D(F)$, iff, $e\in Z(R)$ and $eF(u) =0$ for every $u\in R$.

Keywords- semiprime rings, derivation, generalized reverse derivation.

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1. Introduction

Through our paper, R is ring(associative) and $Z(R)$ denote center R. Recollection, a ring R is semiprime whenever $xRx = (0)$, then $x = 0$. An additive map d: $R\longrightarrow R$ is named a derivation (resp. Jordan derivation) if $d(uv) = d(u)v + ud(v)$ (resp. $d(u^2) = d(u)u + ud(u)$) satisfy for every u, $v\in R$. Following [1] an additive map d : R $\longrightarrow$R named left derivation (resp. Jordan left derivation) if $d(uv) =ud(v) + vd(u)$ (resp. $d(u^2) = 2ud(u)$) for every u, $v\in R$. Obviously, every left derivation on a ring R is a Jordan left derivation, but the converse in general is not true. Ashraf and Ali in [2] defined the generalized left derivation as, an additive map $F :R\longrightarrow R$ is named a generalized left derivation (resp. a generalized Jordan left derivation) related with Jordan left derivation, if there exists a Jordan left derivation $d: R\longrightarrow R$, such that $F(uv) = uF(v) + vd(u)$ (resp. $F(u^2) =uF(u) + ud(u)$) for every u, $v\in R$. In [3] Vukman and Brešar defined the reverse derivation as, an additive map d : R $\longrightarrow$R satisfactory $d(uv) =d(v)u + vd(u)$, for every u, $v\in R$. It is clear, that reverse derivation and derivation are the same if R is commutative. In [4], [5] explored a more reverse derivations. The concept of generalized reverse derivation was introduced first in[6] as, an additive map F : $R\longrightarrow R$ is named a generalized reverse derivation if we have a reverse derivation d : $R\longrightarrow R$, satisfy: $F(uv) =F(v)u + vd(u)$, for every u, $v\in R$. Following Reddy and et al. in [7], work replacing of existence of reverse derivation by derivation in above definition of generalized reverse derivation in [6].The research of dependent elements appeared in [8], by Thaheem and Laradji. Lately, several authors Ali, Chaudhry and others in [9, 10, 11, and 12] proved more results on dependent elements on rings. Let us take $\check{S}:R\longrightarrow R$ as a map, if there exists an element $e\in R$ such that $\check{S}(u)e=[u,e]e$, for every $u\in R$, in this case e is called Generalized Dependent Element of $\check{S}$, and $\check{G}\text{-}D(\check{S})$ denote the set of all Generalized Dependent Elements of $\check{S}$. In this paper the result proved, if R is semiprime ring, and F is a generalized reverse derivation related with derivation d, on R, then $e\in\check{G}\text{-}D(F)$, iff , $e\in Z(R)$ and $eF(u)=0$ for every $u\in R$. In addition, we gave some results with semiprime ring of Generalized Dependent Elements for generalized reverse derivation related with derivation d.

2. The Results

<u>Theorem.1</u>

Suppose that R is semiprime ring, then

1- If there exist $x\in R$, such that $x[x, u]= 0$, for every $u\in R$, then $x\in Z(R)$, [13].

2-If there exist $x\in R$, such that $[x, u]x= 0$, for every $u\in R$, then $x\in Z(R)$, [6].

<u>Theorem.2 [14]</u>

Let R be a semiprime ring, and d be an inner derivation on R, if $[d(u),u]=0$, for every $u\in R$, then $d=0$.

<u>Theorem.3</u>

Suppose that R is a semiprime ring, and $F:R\longrightarrow R$ is a generalized reverse derivation related with derivation d, then $d(u)\in Z(R)$, for every $u\in R$.

Proof: Let us take:

$F(u^2v) = F(v)u^2+ vd(u^2)$, for every $u, v\in R$ (1)

That is:

$F(u^2v)=F(v)u^2+ vd(u)u +vud(u)$, for every $u, v\in R$ (2)

Moreover, let us take:

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$F(u.uv)= F(uv)u + uvd(u)$, for every $u, v\in R$ (3)
That get:
$F(u^2v)=F(v)\ u^2+ vd(u)u + uvd(u)$, for every $u,v\in R$ (4)
Comparing (2) and (4):
$[u, v]d(u) = 0$, for every $u, v\in R$ (5)
Linearizing (6) on u:
$[u, v]d(t)+ [t, v]d(u) = 0$, for every $t ,u, v\in R$ (6)
Putting v= vz in (5) and from (5), we obtain:
$[u, v]zd(u) = 0$, for every $t ,u, v\in R$ (7)
Also, Putting $z=d(t)z[t, v]$ in (7):
$[u, v]d(t)z[t, v]d(u) = 0$, for every $u, v, z, t\in R$ (8)
From (6) and (8), we obtain:
$[u, v]d(t)z[u, v]d(t) = 0$, for every $t ,u, v, z\in R$ (9)
From hypothesis, R is semiprime, obtain:
$[u, v]d(t) = 0$, for every $u, v, t\in R$ (10)
Theorem (2.1)(ii), give $d(t) \in Z(R)$, for every $t\in R$.

Theorem.4

Let R be semiprime ring, and $F:R\longrightarrow R$ is a generalized reverse derivation related with derivation d. Then $F(u)\in Z(R)$, for every $u \in R$.
Proof: our assumption, give:
$F(uv^2)=F(v^2)u + v^2d(u)$, for every $v ,u\in R$ (1)
That is:
$F(uv^2)= F(v)vu + vd(v)u + v^2d(u)$, for every $v ,u\in R$ (2)
Also,
$F(uv^2)= F((uv)v)= F(v)uv + vd(uv)$, for every $v ,u\in R$ (3)
That is:
$F(uv^2)= F(v)uv + vd(u)v + vud(v)$, for every $v ,u\in R$ (4)
By Theorem 2.3, (4) give:
$F(uv^2)= F(v)uv + v^2d(u) + vd(v)u$, for every $v ,u \in R$ (5)
From (5), (2), we acquire:
$F(v)vu= F(v)uv$, for every $v ,u \in R$ (6)
That is:
$F(v)[v, u] = 0$, for every $v ,u \in R$ (7)
Putting u=ru in (7), and using (7), leads to:
$F(v)r[v, u] = 0$, for every $r,v ,u\in R$ (8)
Furthermore, linearizing of relation (7), give:
$F(v)[s, u] + F(s)[v, u] = 0$, for every $s,u, v\in R$ (9)
This implies:
$F(v)[s, u] = - F(s)[v, u]$, for every $s, u, v \in R$ (10)
Now, putting $r =[s, u]rF(s)$ in (8), and using (10), leads to:
$F(s)[v, u]rF(s)[v, u] = 0$, for every $s ,u, v, r\in R$ (11)
Because R is semiprime, (11) gives:
$F(s)[v, u] = 0$, for every $s ,u, v\in R$ (12)
Using Theorem (2.1)(i), we obtain $F(s) \in Z(R)$ for every s in R.

Corollary .5

Let R be semiprime ring, and $F:R\longrightarrow R$ is a generalized reverse derivation F related with derivation d. Then F is a generalized left derivation with Jordan left derivation d.

Definition.6

Let $\check{S}:R\longrightarrow R$ is a map, if there exists an element $e\in R$ such that $\check{S}(u)e= [u,e]e$, holds for every $u\in R$, in this case e is called Generalized Dependent Element of Š, and Ğ-D(Š) denote the set of all Generalized Dependent Elements of Š.

Examples .7

1)
Let $R = \left\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in Z, \text{the set of integer}\right\}$
Define map Š: R ⟶R, as:
$$\check{S}\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$$
Take the element $e = \begin{bmatrix} a1 & 0 \\ 0 & 0 \end{bmatrix} \in R$
It is clear that, e is Generalized Dependent Element of Š.
Note that if we take $e = \begin{bmatrix} a1 & b1 \\ 0 & c1 \end{bmatrix} \in R$, we show that, e is not Generalized Dependent Element of Š.

2)
Let $R\left\{\begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} : a, b \in Z, \text{the set of integer}\right\}$
Define map Ư: R ⟶R, as:
$$Ư\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Take the element $e = \begin{bmatrix} 0 & a1 & b1 \\ 0 & 0 & a1 \\ 0 & 0 & 0 \end{bmatrix} \in R$

It is clear that; e is Generalized Dependent Element of Ư.

Theorem.8

Let R be semiprime ring, and F:R⟶R is a generalized reverse derivation F related with derivation d. Thus e∈ Ğ-D(F), iff, e∈Z(R) and F(u)e=0, for every u∈R

Proof: From hypothesis, e∈ Ğ-D(F), thus:

F(u)e= [u,e]e, for every u∈ R (1)

Putting u =uv in (1), we have:

F(v)ue+vd(u)e= u [v, e] e +[u, e]ve, for every u, v ∈R (2)

By Theorem 2.4, (2) gives:

uF(v)e+vd(u)e=u [v, e] e +[u, e]ve, for every v,u ∈R (3)

Comparing (3) and (1), to obtain:

vd(u)e=[u, e]ve, for every u,v∈ R (4)

Right multiplication of (4) by z, leads to:

vd(u)ez= [u, e]vez, for every u,v, z∈ R (5)

Putting vz instead of v in (4), we get:

vzd(u)e=[u, e]vze, for every z,u,v∈ R (6)

Use Theorem 2.3 in (6), and Subtracting (5) from (6), we get:

vd(u)[z,e]=[u,e]v[z,e], for every z ,u,v∈ R (7)

Left multiplication of (7) by u, leads to:

uvd(u)[z, e]= u[u, e]v[z, e], for every z ,u,v∈ R (8)

Putting v=uv in (7), to obtain:

uvd(u)[z, e]=[u, e]uv[z, e], for every z ,u,v∈ R (9)

From (8) and (9), we get:

u[u, e]v[z, e]= [u, e]uv[z, e], for every z ,u,v∈ R (10)

This implies:

[u,[u, e]] v [z, e] = 0, for every z ,u,v∈ R (11)

Right multiplication of (11) by z, obtain:

[u,[u, e]] v [z, e]z = 0, for every z ,u,v∈ R (12)

Putting v=vz in (11), we have:

[u,[u, e]] vz [z, e] = 0, for every z ,u,v∈ R (13)

Removing (12) from (13), we get:

[u,[u, e]]v[z,[z, e]]=0, for every z ,u, v∈ R (14)

That get:

[u,[u, e]]v[u,[u, e]]=0, for every u,v∈ R (15)

By hypothesis, R is semiprime, implies:

[u,[u, e]]=0, for every u∈ R (16)

Defined the inner derivation Θ: R ⟶R as: Θ(u)=[u, e], is commuting, from Theorem 2.2, implies[u, e]=0, for every u∈ R, then e∈Z(R)

Further, from relation(1),we obtain F(u)e=0 for every u∈R

Conversely, F(u)e=0, for every u∈R (17)

Also, we have e∈Z(R), this get:

[u, e]=0, for every u∈R (18)

Right multiplication of (18) by e, we get:

[u, e] e = 0, for every u∈R (19)

From (17) and (19), one obtains:

F(u)e= 0 = [u, e] e, for every u∈R

Corollary.9

Let R be semiprime ring, and F:R⟶R is a generalized reverse derivation F related with derivation d. Then e∈ Ğ-D(F), if and only if, e∈Z(R) and eF(u)=0, for every u∈R.

Proof: From hypothesis e∈ Ğ-D(F), thus by Theorem 2.8, e∈Z(R) and F(u)e=0

So, e∈Z(R) ⊂ R, that is e∈R, and by Theorem 2.4 (F(R)⊆ Z(R)), we obtain eF(u)=0, for every u∈R. (we can say e∈Z(R), F(R)⊆ R, we obtain eF(u)=0, for every u∈R)

Conversely, let eF(u)=0 , by Theorem 2.4 (F(R)⊆ Z(R)), implies

F(u)e=0, for every u∈R (1)

And e∈Z(R), that is:

[u, e]=0, for every u∈R (2)

Right multiplication of relation (2) by e, we get:

[u, e] e = 0, for every u∈R (3)

From (1) and (3), one obtains:

F(u)e=0= [u, e] e, for every u∈R (4)

This implies e∈ Ğ-D(F).

Corollary .10

Let R be semiprime ring, and F:R⟶R is a generalized reverse derivation F related with derivation d, if e∈ Ğ-D(F), then d(e)=0

Proof: From hypothesis, e∈ Ğ-D(F),thus by Corollary 2.9, we obtain:

eF(u)=0, for every u∈ R (1)

Putting u=ru in (1), one obtains:

$eF(u)r + eud(r) = 0$, for every $u, r \in R$ (2)
From (1) and (2), one obtains:
$eud(r) = 0$, for every $u, r \in R$ (3)
Putting $d(r)ue$ instead of u in (3), get:
$ed(r)ue\ d(r)=0$, for every $u, r \in R$ (4)
From hypothesis, get:
$e\ d(r)=0$, for every $r \in R$ (5)
Putting $r = d(r)$ in (5), get:
$e\ d(d(r))=0$, for every $r \in R$ (6)
Also from(5),we get:
$d(e\ d(r))=0$, for every $r \in R$ (7)
This gets:
$d(e)d(r)+ed(d(r))=0$, for every $r \in R$ (8)
From (6) and (8), we get:
$d(e)d(r)=0$, for every $r \in R$ (9)
Putting re instead r in (9), get:
$d(e)d(r)e + d(e)rd(e)=0$, for every $r \in R$ (10)
From (9) and (10), we get:
$d(e)rd(e)=0$, for every $r \in R$ (11)
Again, R is semiprime, implies:
$d(e)=0$ (12)

Corollary.11

Let R be semiprime ring, and $F:R \longrightarrow R$ is a generalized reverse derivation F related with derivation d, if $e \in$ Ğ-D(F), then $F(e)=0$

Proof: From hypothesis $e \in$ Ğ-D(F),thus by Corollary 2.9, get:
$eF(u)=0$, for every $u \in R$ (1)
Also, Theorem 2.8, get:
$F(u)e=0$, for every $u \in R$ (2)
From (2), implies:
$F(F(u)\ e)=0$, for every $u \in R$ (3)
This implies:
$F(e)F(u)+ed(F(u))=0$, for every $u \in R$ (4)
Also, From (1),
$d(eF(u))=0$, for every $u \in R$ (5)
This implies:
$d(e)F(u)+ed(F(u))=0$, for every $u \in R$ (6)
From Corollary 2.10:
$ed(F(u))=0$, for every $u \in R$ (7)
From (4) and (7), get:
$F(e)F(u)=0$, for every $u \in R$ (8)
Right multiplication of relation(8) by r, leads to:
$F(e)F(u)r=0$, for every $u, r \in R$ (9)
Theorem 2.4, leads to:
$F(e)\ r\ F(u)=0$, for every $u, r \in R$ (10)
For u=e in (10) ,implies:
$F(e)\ r\ F(e)=0$, for every $r \in R$ (11)
By hypothesis of R implies:
$F(e)=0$ (12)

Corollary .12

Let R be semiprime ring, and $F:R \longrightarrow R$ is a generalized reverse derivation F related with derivation d, then Ğ-D(F) is semiprime commutative subring of R.

Proof: take $e \in$ Ğ-D(F),thus Theorem 2.8, gives:
$e \in Z(R)$ (1)
This get:
$eu=ue$, for every $u \in R$ (2)
Also, let $b \in$ Ğ-D(F), by Theorem 2.8
$b \in Z(R)$ (3)
That is:
$bu=ub$, for every $u \in R$ (4)
Subtracting (4) from (2), obtain:
$(e-b)u=u(e-b)$, for every $u \in R$ (5)
This get:
$(e-b) \in Z(R)$ (6)
Since the element e, and the element $b \in$ Ğ-D(F), and from Theorem 2.8, we get:
$F(u)\ e=0$,
And
$F(u)\ b=0$, for every $u \in R$ (7)
That is:
$F(u)\ (e-b)=0$, for every $u \in R$ (8)
From (6), (8), and Theorem 2.8, we get:
$(e-b) \in$ Ğ-D(F) (9)
Now since $b \in$ Ğ-D(F), by Corollary 2.9, we get:
$b\ F(u)=0$, for every $u \in R$ (10)
Multiply of (10) from the left by a, obtain:
$eb\ F(u)=0$, for every $u \in R$ (11)
Right multiplication of (2) by b, we get:
$eub=ueb$, for every $u \in R$ (12)
From (4), (12), we get:
$ebu=ueb$, for every $u \in R$ (13)

That is:

eb∈Z(R) (14)

By Corollary 2.9, and (11) and (14), gives:

eb∈ Ğ-D(F) (15)

From (9) and (15), we get Ğ-D(F) subring of R.

Also, because e, b∈Z(R) ⊂ R, we get eb= be, that is Ğ-D(F) is commutative subring of R.

To prove Ğ-D(F) is semiprime, take eĞ-D(F)e = 0 ,e∈ Ğ-D(F), Then:

eue = 0, for every u∈ Ğ-D(F)

In particular, $e^3=0$

Thus we get e =0 (R is semiprime)

That is, Ğ-D(F) is semiprime.

Corollary.13

Let R be commutative semiprime ring, and F:R⟶R is a generalized reverse derivation F related with derivation d. then Ğ-D(F) is an ideal of R.

Proof: take e, b∈ Ğ-D(F), and from (9) in Corollary 2.12 (see (1) to (9) in Corollary 2.12), we get:

(e-b)∈ Ğ-D(F) (1)

Also, take e∈ Ğ-D(F), and take r∈R, then:

F(u)e=0, for every u∈ R (2)

Right multiply of (2) from the right by r, to get:

F(u)er=0, for every u∈ R (3)

Also, Since e∈Z(R), r∈R , then:

er =re (4)

From (3),(4), obtain:

F(u)er= F(u)re =0, for every u∈ R (5)

Now, Since R commutative:

er∈Z(R) (6)

From (5), (6), and by Theorem 2.8, we get:

er = re∈ Ğ-D(F) (7)

From (1), (7), we get Ğ-D(F) ideal

Remark .14:

Let R be semiprime ring, I is an ideal on R, then:

(i) I is semiprime subring of ring R

(ii)The center of I is contained in center of R.

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