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The Approximate Solution of the Fornberg-Whitham Equation by a Semi-Analytical Iterative Technique

Abstract- *In this work, two semi-analytical methods are introduced in order to handle the solution of the Fornberg-Whitham equation. The first one method has been proposed by Temimi and Ansari, namely the TAM. The second method is the Banach contraction method which is briefly called the BCM. Both methods do not require using any additional assumptions. Our calculations are distinguished by an efficiency and rapidity of obtaining the results, in comparison with the previous studies for solving the same problem.*

Keywords- *Fornberg-Whitham equation; Temimi-Ansari method; Banach contraction method; numerical solution.*

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1. Introduction

It is clear that various important phenomena occurring in many scientific and geometric fields of science, and engineering are often modeled to linear and nonlinear equations. Of these types are models with nonlinear partial differential equations that can help people to identify the deep processes described.

The Fornberg Whitham type equation [1] is given as

$$f_t - f_{xxt} + f_x = f f_{xxx} - f f_x + 3 f_x f_{xx}, \qquad (1)$$

as it has other types of wave solutions which are called kink-like and anti kink-like which have not yet been evaluated for this type of equations [2]. The traveling wave solution is an important type of solutions for nonlinear PDEs. Over time, several modern methods have been found to solve this type of this equation.

In this work, two iterative techniques are used for obtaining the solution of the Fornberg-Whitham equation (1) easily. The first iterative method has been presented by Temimi and Ansari [2] in 2011, called the TAM. The TAM has been recently used for solving many problems [3-8] in different specializations. The second iterative method for solving the same problem is the Banach contraction method (BCM) where it is an iterative technique based on using Banach contraction principle [9]. The BCM has been presented by Daftardar-Gejji and Bhalekar in 2009.

The approximate solution of these used methods will be discussed numerically and the results will be reviewed as it has been studied previously in the other several methods [10-14].

In this study, the main ideas of the used two iterative methods are presented in section 2. In section 3, application of the TAM and BCM are given for solving the current problem. Numerical results for our solutions are offered in section 4. The conclusion is presented in the last section.

2. The Basic Ideas for the Iterative Methods

In this section, we have reviewed the analysis of the two used iterative techniques.

I. Basic idea of the TAM

For verifying the standard concepts of the TAM [2], Let begin with the following form

$$L\big(f(x,t)\big) + N\big(f(x,t)\big) - g(x,t) = 0, x \in \mathbb{D}, t > 0 \qquad (2)$$

with the boundary conditions

$$B\left(f, \frac{\partial^i f}{\partial x^i \partial t^i}\right) = 0, i = 0,1,2,\dots,k.\ k \in N \qquad (3)$$

The first step in the TAM is finding the initial approximation namely the function $f_0(x,t)$. This function represents the solution for the initial problem

$$L\big(f_0(x,t)\big) - g(x,t) = 0, \qquad (4)$$

with

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$$B\left(f_0, \frac{\partial^i f_0}{\partial x^i \partial t^i}\right) = 0 \quad i = 0,1,2,\ldots,k. \quad (5)$$

The following step is evaluating the next iterative approximations. So, for finding $f_1(x,t)$; we must find the solution for the following problem

$$L(f_1(x,t)) + N(f_0(x,t)) - g(x,t) \quad 0, \quad (6)$$

$$B\left(f_1, \frac{\partial^i f_1}{\partial x^i \partial t^i}\right) = 0,\ i = 0,1,2,\ldots,k. \quad (7)$$

The general formula for the evaluating the TAM approximations is given as:

$$L(f_{n+1}(x,t)) + N(f_n(x,t)) - g(x,t) = 0 \quad (8)$$

With

$$B\left(f_{n+1}, \frac{\partial^i f_{n+1}}{\partial x^i \partial t^i}\right) = 0, i = 0,1,2,\ldots,k. \quad (9)$$

Continuing in this manner, we can get an appropriate function which converges to the exact solution for equation (2).

II. Basic idea of the BCM

The following basic concepts [9] for this method have been introduced.

1. Preliminaries for the BCM

Definition 1 Let M_1 and M_2 be two metric spaces, and let F be a mapping from M_1 into M_2, F is said to be Lipschitz mapping if there exists some real number $r \geq 0$ such that for all $x_1, x_2 \in M_1$ we have $d(Fx_1, Fx_2) \leq r d(x_1, x_2)$. F is said to be a contraction mapping if $r < 1$ [9].

Theorem 1 [9] **(Banach contraction principle)** Consider a contraction mapping $F: M \to M$ with some z which is a Lipschitz constant, where M is a complete metric space; then F has some unique fixed point f in the space M. In addition, if there is some arbitrary point x_0 in M, and x_n which is given as $x_{n+1} = F(x_n),\ n = 0,1,2,\ldots$ so $\lim_{n\to\infty} x_n = f$ and $d(x_n, f) \leq \frac{r^n}{1-r} d(x_1, x_0)$.

Theorem 2 [9] Let F be a mapping of some complete metric space M into the same space, such that F^k is the contraction mapping of M for some positive integer k, then F has some unique fixed point in this space M.

III. Basic steps of the BCM:

Consider the following nonlinear PDE:

$$f(x,t) = g(x,t) + N[f(x,t)], \quad (10)$$

where $f(x,t)$ is an unknown function which is the main goal, $g(x,t)$ is just a given function and N represents a nonlinear operator for the functional equation (10). Now we have to define some successive approximations as the following:

$$f_0 = g,$$
$$f_1 = f_0 + N[f_0], \quad (11)$$
$$f_2 = f_0 + N[f_1],$$
$$\vdots$$
$$f_n = f_0 + N[f_{n-1}],$$
$$n = 1,2,\ldots. \quad (12)$$

Since the BCM depends on the Banach contraction principle in finding new approximations and according to Theorem 2; the mapping N of the nth sequence which is given in (12) will be convergent to the exact solution by:

$$f = \lim_{n\to\infty} f_n. \quad (13)$$

3. The Application of the Methods

In what follows, the solution for equation (1) with the initial condition

$$f(x,0) = e^{\frac{1}{2}x}, \quad (14)$$

by using the TAM and BCM will be presented. The exact solution for this problem is given as [11]

$$f(x,t) = e^{\frac{1}{2}x - \frac{2}{3}t}, \quad (15)$$

I. Applying the TAM

Let us consider the FW equation (1), assuming the initial condition (14). To find the first approximation f_0, we solve the following initial problem:

$$\partial_t f_0 = 0,$$
$$f_0(x,0) = e^{\frac{1}{2}x}, \quad (16)$$

we have

$$f_0(x,t) = e^{\frac{1}{2}x}. \quad (17)$$

The next iteration f_1 can be obtained by calculating the following problem:

$$\partial_t f_1 = f_{0xxt} - f_{0x} + f_0 f_{0xxx} - f_0 f_{0x} + 3 f_{0x} f_{0xx},$$
$$f_1(x,0) = e^{\frac{1}{2}x}, \quad (18)$$

we get:

$$f_1 = e^{x/2} - \frac{1}{2} e^{x/2} t. \quad (19)$$

Subsequently, the next iterations f_2, f_3 and f_4 will be found by solving the following problems:

$\partial_t f_2 = f_{1xxt} - f_{1x} + f_1 f_{1xxx} - f_1 f_{1x} + 3 f_{1x} f_{1xx},\ f_2(x,0) = e^{\frac{1}{2}x},$
$\partial_t f_3 = f_{2xxt} - f_{2x} + f_2 f_{2xxx} - f_2 f_{2x} + 3 f_{2x} f_{2xx},\ f_3(x,0) = e^{\frac{1}{2}x},$
$\partial_t f_4 = f_{3xxt} - f_{3x} + f_3 f_{3xxx} - f_3 f_{3x} + 3 f_{3x} f_{3xx},\ f_4(x,0) = e^{\frac{1}{2}x},$
(20)

we get the following approximations:

$$f_2 = e^{x/2} - \frac{5}{8} e^{x/2} t + \frac{1}{8} e^{x/2} t^2, \quad (21)$$

$$f_3 = e^{x/2} - \frac{21}{32} e^{x/2} t + \frac{3}{16} e^{x/2} t^2 - \frac{1}{48} e^{x/2} t^3, \quad (22)$$

$$f_4 = e^{x/2} - \frac{85}{128} e^{x/2} t + \frac{27}{128} e^{x/2} t^2 - \frac{7}{192} e^{x/2} t^3 + \frac{1}{384} e^{x/2} t^4. \quad (23)$$

For finding f_n, we can solve the problem in the genarlized formla

$$\partial_t f_{n+1} = f_{nxxt} - f_{nx} + f_n f_{nxxx} - f_n f_{nx} + 3 f_{nx} f_{nxx},\ f_{n+1}(x,0) = e^{\frac{1}{2}x}. \quad (24)$$

II. Applying the BCM

In order to solve the FW equation given in (1) subjected to the initial condition (14) by the BCM, let start by integrating both sides of equation (1) with respect to t. we can obtain the integral formula:

$$f(x,t) = e^{\frac{1}{2}x} + \int_0^t (f_{xx\tau} - f_x + f f_{xxx} - f f_x + 3 f_x f_{xx})\, d\tau. \quad (25)$$

Let define the following recurrence scheme

$$f_0(x,t) = e^{\frac{1}{2}x}, \quad (26)$$

$$f_1(x,t) = f_0 + \int_0^t (f_{0xx\tau} - f_{0x} + f_0 f_{0xxx} - f_0 f_{0x} + 3 f_{0x} f_{0xx})\, d\tau, \quad (27)$$

and in general, we have

$$f_{n+1}(x,t) = f_0 + \int_0^t (f_{nxx\tau} - f_{nx} + f_n f_{nxxx} - f_n f_{nx} + 3 f_{nx} f_{nxx})\, d\tau, n \geq 1. \quad (28)$$

Thus, we get the following approximations

$$f_1 = e^{x/2} - \frac{1}{2} e^{x/2} t,$$

$$f_2 = e^{x/2} - \frac{5}{8} e^{x/2} t + \frac{1}{8} e^{x/2} t^2,$$

$$f_3 = e^{x/2} - \frac{21}{32} e^{x/2} t + \frac{3}{16} e^{x/2} t^2 - \frac{1}{48} e^{x/2} t^3,$$

$$f_4 = e^{x/2} - \frac{85}{128} e^{x/2} t + \frac{27}{128} e^{x/2} t^2 - \frac{7}{192} e^{x/2} t^3 + \frac{1}{384} e^{x/2} t^4,$$

$\vdots$

and so on. The approximate solutions obtained by the BCM and the TAM are equal.

4. The Numerical Analysis

In Table 1, the computation of the absolute errors for the differences between the exact solution (15) and the fourth approximate solution (23) obtained by the TAM (or BCM). The obtained numerical solutions are similar to the previous numerical HAM and ADM results [11]. Furthermore, Figures 1 and 2 show the approximate and the exact solutions for the FW problem respectively. Figure 3 shows a comparison between the exact and approximate solutions obtained by the TAM (or BCM).

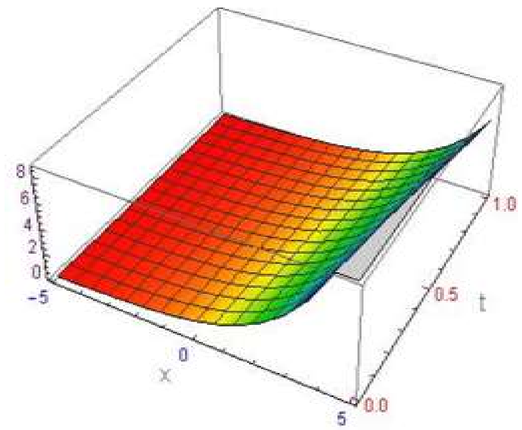

Figure 1: A 3D plot for the 4th order approximate solution obtained by the TAM (or BCM) of the Fornberg-Whitham problem

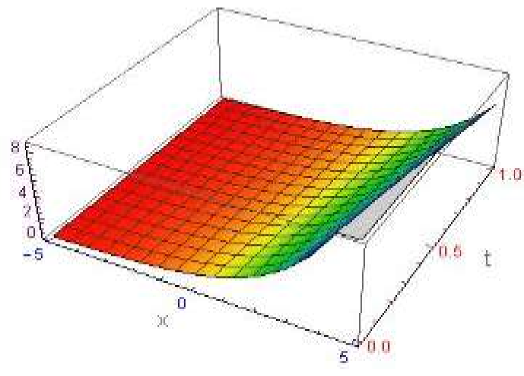

Figure 2: A 3D plot for the exact solution of the Fornberg-Whitham problem

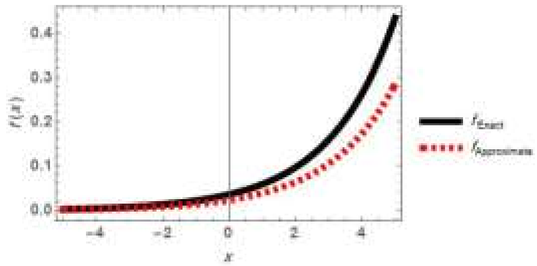

Figure 3: A comparison between the exact and 4th approximate solution at t = 5

Table 1: The numerical values for the exact and the 4th order approximate solutions with the absolute errors at t=5

x_i	f_{Exact}	$f_{Approximate}$	Absolute error
-4	2.89083×10^{-11}	1.85053×10^{-9}	2.10815×10^{-8}
-2	2.88097×10^{-11}	1.84491×10^{-9}	2.1025×10^{-8}
0	2.86402×10^{-11}	1.83468×10^{-9}	2.0916×10^{-8}
2	2.8398×10^{-11}	1.81987×10^{-9}	2.07547×10^{-8}
4	2.80852×10^{-11}	1.80051×10^{-9}	2.05416×10^{-8}

5. Conclusion

In this study, two modified iterative methods have been used for solving the Fornberg-Whitham problem where the appropriate results are obtained in accordance with the other methods used previously. Our work is characterized by the fact that it does not require using of any restricted assumptions in solving process, i.e., using Adomian polynomials in the ADM, Lagrange multiplayer in the VIM, the auxiliary things in the HAM, or any other additional parameters. This makes our work characterized by quickness, efficiency, and it has the ease in producing approximate solutions.

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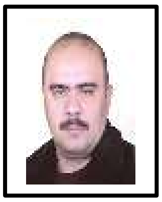
Sinan Hatif, born in 1976, received the BS degree in Mathematics from the University of Al-mustansiriya in 1998 and MS Degree in Applied Mathematics Baghdad-College of Education for Pure Science Ibn-Al-Haitham,Department of Mathematics in 2002. His main interests are analytic, iterative and numerical methods for integral equations, ordinary differential equations and partial differential equations. Currently,



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