STRUTS CARRYING NON – UNIFORM AXIAL LOAD ON ELASTIC FOUNDATION

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Scientific Journal
The University of Technology
Baghdad

Vol. 2
No. 1
December 1978

https://doi.org/10.30684/etj.1978.1.4
2412-0758/University of Technology-Iraq, Baghdad, Iraq

الركائز الحاملة لاثقال محورية متغيرة والمحمولة على اسس مرنة

بقلم
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خلاصة المقالة :

يقدم البحث جداول لعدم الاستقرار المحورة والتي تستعمل في تقدير الاثقال الحرجة لهياكل واطر حاوية على ركائز محملة باثقال محورية متغيرة والتي تتغير طوليا من صفر في احدى النهايتين والمحمولة على او المزروقة في تربة مرنة ذات معامل مقاومة هبوط ثابت باستعمال حسابات يدوية مبسطة .

ABSTRACT

The paper presents tables of modified stability functions. They enable the prediction of the elastic critical loads of structures with some members carrying linearly varying axial forces with zero axial force at one end. supported by or driven into elastic soil. having constant modulus of subgrade, using a hand computing method.

INTRODUCTION

Stability functions for beam- column supported on continuous Winkler foundation or driven into elastic soil. carrying linearly varying axial force with zero axial force at one end. are tabulated for different values of axial load parameters and soil stiffnesses. They enable the prediction of the elastic critical loads of structures with some members supported by or driven into elastic soil. having constant modulus of subgrade reaction. using a hand computing method. Applicable to friction piles with full or partial penetration. the upper chords of trusses and other related structures with any ends restraints.

MODIFIED SLOPE DEFLECTION EQUATIONS

The modified slope deflection equations associated with the elastic strut supported by an elastic foundation and carrying a constant axial force P. are given in reference 1. and shown in Fig. 1.

$$M_{12} = (EI/L)(s\theta_1 + sc\theta_2 + Qy_1/L - qQ\,y_2/L) \quad \text{........(1a)}$$

$$M_{12} = (EI/L)(sc\theta_1 + s\theta_1 + qQ\,y_1/L - Q\,y_2/L) \quad \text{........(1b)}$$

$$V_{12} = (EI/L^2)(Q\theta_1 + qQ\,\theta_2 + T\,y_1/L - Tt\,y_2/L) \quad \text{........(1c)}$$

$$V_{21} = (EI/L^2)(qQ\,\theta_1 + Q\,\theta_2 + Tt\,y_1/L - T\,y_2/L) \quad \text{........(1d)}$$

Where

M_{12} is the clockwise end moment at end 1

V_{12} is the end shear force at end 1

θ_1 is the clockwise rotation of end 1

y_1 is the displacement of end 1 perpendicular to member 12

E Young Modulus

I Moment of inertia

L Length of the member 12

s is the modified stiffness factor of the strut

sc is the modified moment carry-over factor

Q is the modified flexural shear stiffness factor

qQ is the modified flexural shear carry-over factor

T is the modified shear stiffness factor

Tt is the modified shear carry-over factor

The modified stability functions are functions of the non-dimensional parameters $p = P/P_e$ and $\lambda L = (K/4EI)^{\frac{1}{4}} L$ where P_e is the Euler load $\pi^2 EI/L^2$ and k is the stiffness of the elastic soil which is equal to the modulus of the subgrade reaction of the soil times the width of the strut. The modified stability functions were determined and graphed against p for different values of λ L.

When the axial load is linearly varying as shown in Fig. 2, the modified stability functions of the two ends differ and the modified slope deflection equations become:

$$M_{12} = (EI/L)(s_1\theta_1 + sc\,\theta_2 + Q_1 y_1/L - (qQ)_1 y_2/L) \qquad \text{........(2a)}$$

$$M_{21} = (EI/L)(sc\,\theta_1 + s_2\theta_2 + (qQ)_2 y_1/L - Q_2 y_2/L) \qquad \text{........(2b)}$$

$$V_{12} = (EI/L^2)(Q_1\theta_1 + (qQ)_2\theta_2 + T_1 y_1/L - Tt\, y_2/L) \qquad \text{........(2c)}$$

$$V_{21} = (EI/L^2)((Qq)_1\theta_1 + Q_2\theta_2 + Tt\, y_1/L - T_2 y_2/L) \qquad \text{........(2d)}$$

where s_1, sc, Q_1, $(qQ)_1$, Tt and T_1 are the modified stability functions of end 1 and are functions of $p = P/2P_e$ and $\lambda L = (k/4EI)^{\frac{1}{4}} L$.

MODIFIED STABILITY FUNCTIONS DERIVATION

The elastic stability of a uniform bar, a member of a frame, subjected to a linearly varying axial load, shown in Fig. 2, on elastic foundation having a stiffness k depend on the solution of the basic differential equation[2]

$$EI\frac{d^4z}{dx^4} + \frac{d}{dx}\left(\frac{Px}{L}\frac{dz}{dx}\right) + kz = 0 \qquad \text{........(3)}$$

where z is the deflection at distance x along the strut. Equation (3) have solutions depending on the parameter k. When k = o, the solution[3] is

$$z = D_1\int^{wx} A_i(t)dt + D_2\int^{wx} B_i(t)dt - \pi V\int^{wx} G_i(t)dt + D_3 \qquad \text{........(4)}$$

where

$w^3 = (P/EIL)$

$A_i(x)$, $B_i(x)$, $G_i(x)$ are Airy integral functions defined by

$$A_i(x) = \frac{1}{\pi}\int_0^{x} \cos(\tfrac{1}{3}t^3 + xt)\,dt$$

$$B_i(x) = \frac{1}{\pi}\int_0^{x} (\sin(\tfrac{1}{3}t^3 + xt) + e^{(-\frac{1}{3}t^3 + xt)})\,dt$$

$$G_i(x) = \frac{1}{\pi}\int_0^{x} \sin(\tfrac{1}{3}t^3 + xt)\,dt$$

D_1, D_2, D_3 are constants of integration

V constant end sheat force

The modified stability functions are also function of the Airy integral functions. They were tabulated for different values of axial load parameters $p = P/2P_e$.

For $k > o$ no closed analytical solution of equation (3) is avaliable. A finite difference method appears to offer the most practical approach to the solution of equation (3).

Equation (3) may be written in difference form[4] as

$$z_{n-2} + A_n z_{n-1} + B_n z_n + C_n z_{n+1} + z_{n+2} = 0 \quad \text{........(5)}$$

The first, second, third and fourth derivative of z is approximated, using the central difference, by.

$$\left(\frac{dz}{dx}\right)_n \approx \frac{1}{2h}(z_{n+1} - z_{n-1}) \quad \text{........(6a)}$$

$$\left(\frac{d^2z}{dx^2}\right)_n \approx \frac{1}{h^2}(z_{n+1} - 2z_n + z_{n-1}) \quad \text{........(6b)}$$

$$\left(\frac{d^3z}{dx^3}\right)_n \approx \frac{1}{2h^3}(z_{n+2} - 2z_{n+1} + 2z_{n-1} - z_{n-2}) \quad \text{....(6c)}$$

$$\left(\frac{d^4z}{dx^4}\right)_n \approx \frac{1}{h^4}(z_{n+2} - 4z_{n+1} + 6z_n - 4z_{n-1} + z_{n-2}) \quad \text{........(6d)}$$

where

N number of increments

$h = L/N$

$$A_n = \frac{\pi^2}{N^3} p(n - 0.5) - 4 \quad \text{........(7a)}$$

$$B_n = 6 - 2n\pi^2 p/N^3 + 4(\lambda L/N)^4 \quad \text{........(7b)}$$

$$C_n = (\pi^2 p/N^3)(n + 0.5) - 4 \quad \text{........(7c)}$$

$p = P/2P_e$

$\lambda L = (k/4EI)^{\frac{1}{4}} L$

The subdivision of the strut into N increments is shown in Fig. 3. Equation (5) give N equations with (N + 4) deflections unknowns. The additional equations are obtained from the boundary conditions at the ends, given in equations (8)

At $x = nh = o$

$z = y_1$

$(dz/dx) = \theta_1$

At $x = Nh = L$

$z = y_2$

$(dz/dx) = \theta_2$ ....(8)

The modified stability functions are obtained from relations given in equations (9) and (2)

At $x = n h = o$ $\quad$ $x = N h = L$

$$\frac{d^2z}{dx^2} = -(M_{12}/EI) \qquad \frac{d^2z}{dx^2} = (M_{21}/EI)$$

$$\frac{d^3z}{dx^3} = (1/EI)(-V_{12}) \qquad \frac{d^3z}{dx^3} = (1/EI)(V_{21} + P\,\theta_2) \qquad \text{........(9)}$$

A finite element approach for estimating the modified stability functions for a stepped varaiation in the soil stiffness or in the axial force using equations (1) is possible. This is being investigated at the University of Technology Baghdad under the supervision of the writer.

STABILITY FUNCTIONS TABLES

To facilate the estimation of the elastic critical load of structures with some member on elastic foundation, it was necessary to tabulate the stability functions. The modified stability functions were tabulated for different values of p for $\lambda L = 1,2,3,4,5$ in Tables 1,2,3,4,5. N is taken to be 41. Bolton and Al-Sarraf[5] Tables are for the special case when $\lambda L = o$.

The stability functions obtained by the finite difference method for $\lambda L = 0$ and $p \geqq 0$ are compared to those obtained by the Airy integral functions and are shown in Fig. 4 and for $p = 0$ and $\lambda L = 0$ are compared to those obtained by the hyperbolic and trigonomentrie functions and are shown in Table 6. The inaccuracy in the modified stability functions are unlikely lead to an inaccuracy in the elastic critical load of structures of the order of less than five percent.

EXAMPLE 1

The slastic critical load of the strut shown in Fig. 5 pinned at A and A' and subjected to a linearly varying axial force, which may represent the upper chord members of a truss, is predicted.

The method of finding the elastic critical load is to assume a value of the axial load parameter p, hence allowing values of the stability functions to be obtained from tables. An infinitesimal disturbance is now applied to the struture and the resistance to this disturbance is found. If this resistance is positive, the structure is not unstable and a higher value of p is tested. If there is zero resistance, the structure is at its critical load and the correct value of p has been found. If the resistance to the disturbance is negative, the structure is unstable, and a lower value of p is tested.

For the strut of Fig. 5, there are two possible modes of elastic instability. The first mode is the non-sway joint rotation mode in which the obvious testing disturbance to use is a rotation of joint B by θ_B and joints A and A' rotated by $\theta_{A'}$. Equations (2a,b) become.

$$M_{AB} = (EI/L)(s_1\theta_A + sc\,\theta_B) \qquad \text{........(10a)}$$

$$M_{BA} = (EI/L)(sc\,\theta_A + s_2\theta_B) \qquad \text{........(10b)}$$

since y_A and y_B are equal to zero. At the critical load $M_{AB} = M_{BA} = 0$ and quations (10a,b) give

$$s_1\,\theta_A + sc\,\theta_B = 0 \qquad \text{........(11a)}$$

$$sc\,\theta_A + s_2\,\theta_B = 0 \qquad \text{........(11b)}$$

and the determinant of the coefficients equated to zero give

$$K = \begin{vmatrix} s_1 & sc \\ sc & s_2 \end{vmatrix} = s_1 s_2 - (sc)^2 = 0 \qquad \text{........(12)}$$

The load parameter p making K vanish is the critical load parameter for the non-sway joint rotation mode which have different values for different λ L.

The second mode is the sway mode and the testing disturbance to use is a displacement of B by y_B and joints A and A' rotate by θ_A but in opposite directions. Equations (2a,d) become.

$$M_{AB} = (EI/L)(s_1 \theta_A - (qQ)_1 y_B/L) \qquad \text{........(13a)}$$

$$V_{BA} = (EI/L^2)((qQ)_1 \theta_A - T_2 y_B/L) \qquad \text{........(13b)}$$

since θ_B and y_A equal zero. At the critical load $M_{AB} = V_{BA} = 0$ and equations (13a,b) give

$$s_1 \theta_A - (qQ)_1 y_B/L = 0 \qquad \text{........(14a)}$$

$$-(qQ)_1 \theta_A + T_2 y_B/L = 0 \qquad \text{........(14b)}$$

and the determinant of coefficients of unknowns equated to zero give

$$K = \begin{vmatrix} s_1 & -(qQ)_1 \\ -(qQ)_1 & T_2 \end{vmatrix} = s_1 T_2 - (qQ)_1^2 = 0 \qquad \text{........(15)}$$

Solution of equation (15) yields the critical load parameter p of the strut for the sway mode. The elastic critical loads of the two modes of the strut will be estimated for λ L = 3.

Non sway mode

First trial p = 3.0

From Table 3 s_1 = 4.955 sc = 0.565 s_2 = 0.230; substituting these values in equation (12) yields.

K = 4.955 × 0.230 − $(0.565)^2$ = + 0.822 i.e the strut is stable and a higher of p is tried.

Second trial p = 3.25, when p = 3.25, the value of K is found to be − 3.037 i.e unstable, hence the critical p by linear interpolation is p = 3.00 + (3.25 − 3.00) (0.822) / (0.822 + 3.037) = 3.053

sway mode

First trial p = 3.00

From Table 3 s_1 = 4.955 $(qQ)_1$ = 7.024 T_2 = 24.076; substituting these values in equation (15) yields

K = 4.955 × 24.076 − $(7.024)^2$ = + 69.95 i.e the strut is stable and a higher value of p is tried.

Second trial p = 3.25, when p = 3.25, the value of K is - 6.542 and the critical p is 3.229.

Thus the non-sway mode for the strut is the critical mode and the critical load of the strut is

$p = (P/2P_e) = 3.053$

$\therefore P = 2 \times 3.053 \times (\pi^2 EI / (L_{AA'}/2)^2) = 24.426 (\pi^2 EI / L^2_{AA'})$

and the corresponding effective length of the strut is

$L_e = (L_{AA'} / \sqrt{24.426}) = 0.202 L_{AA'}$

EXAMPLE 2

The elastic critical load of the frame shown in Fig. 6 is found when only the stanchions AB and A'B' are supported laterally by elastic soil and the axial forces are assumed to be linearly varying.

To find the elastic critical load of the sway mode, joint B is translated laterally by δ_B and joint B and B' rotate by θ_B. The end moments and forces of the member due to these deformations using equations (2b.d) are

$$M_{BA} = (EI/L)_1 (s_2 \theta_B - Q_2 \delta_B / L_1) \quad \text{.......(16a)}$$

$$M_{BB'} = (EI/L)_2 (6 \theta_B) \quad \text{.......(16b)}$$

$$V_{BA} = (EI/L^2)_1 (Q_2 \theta_B - T_2 \delta_B / L_1) \quad \text{.......(16c)}$$

where the suffixes 1 and 2 refer to stanchion AB and beam BB' respectively and no axial load in the beam.

joint equilbrium at B require that

$$\begin{vmatrix} \Sigma M_B \\ \\ \Sigma V_B \end{vmatrix} = \begin{vmatrix} (EI/L)_1 s_2 + 6(EI/L)_2 & -(EI/L)_1 Q_2 \\ \\ -2(EI/L^2)_1 Q_2 & +2(EI/L^2)_1 T_2 \end{vmatrix} \begin{vmatrix} \theta_B \\ \\ S_B/L \end{vmatrix} \quad \text{.......(17)}$$

At the critical load $\Sigma M_B = \Sigma V_B = 0$ and equation (17) gives

$$K = \begin{vmatrix} s_2 + 6(EI/L)_2/(EI/L)_1 & -Q_2 \\ \\ -Q_2 & T_2 \end{vmatrix} = 0 \quad \text{.......(18)}$$

When L and EI are assumed to be constant throughout, the elastic load of the frame is given by the equation

$$K = \begin{vmatrix} s_2 + 6 & -Q_2 \\ \\ -Q_2 & +T_2 \end{vmatrix} = 0 \quad \text{.......(19)}$$

and is obtained by trial and interpolation. The elastic critical load parameter p of the frame will be estimated for $\lambda L = 5$.

First trial $p = 4.5$

From Table 5 $s_2 = 5.090$ $\quad Q_2 = 53.397$ $T_2 = 273.075$

Substituting these values in equation (19) yields

$$K = \begin{vmatrix} 11.090 & -53.396 \\ -53.397 & 273.075 \end{vmatrix} = +177.2 \qquad \text{i.e. stable}$$

Second trial p = 5.0, which yields K = − 1097 and the critical load parameter p is 4.64.

If joint B is held in position, the sway factors in equation (19) vanish and the stability criterion becmes

$$K = s_2 + 6 = 0 \qquad \text{.......(20)}$$

The load parameter p satisfying equation (20) for $\lambda L = 5$ is found to be 8.95. Thus the sway mode is the critical one.

CONCLUSIONS

1. The method permits the rapid determination of elastic critical loads of structures having some members on or driven into elastic soil having constant modulus of subgrade reaction and the axial forces are linearly varying from zero axial force at one end to P at the other end.

2. Tables of elastic stability functions are tabulated against $p = P/2P_e$ for $\lambda L = 1,2,3,4,5$ that will be usefull for any of the methods developed for the prediction of the elastic critical loads.

Appendix 1: REFERENCES

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TABLE 1 $\lambda L = 1$

p	s_1	sc	s_2	Q_1	$(qQ)_1$	$(qQ)_2$	Q_2	T_1	Tt	T_2
0	4.034	1.968	4.034	6.201	5.870	5.870	6.201	13.465	11.476	13.465
0.1	3.968	2.001	3.836	6.005	5.670	5.869	6.206	12.289	10.289	12.266
0.2	3.900	2.037	3.632	5.805	5.464	5.870	6.214	11.101	9.090	11.055
0.3	3.830	2.074	3.422	5.599	5.253	5.875	6.226	9.901	7.878	9.831
0.4	3.757	2.115	3.204	5.388	5.037	5.883	6.241	8.688	6.653	8.592
0.5	3.681	2.159	2.980	5.171	4.814	5.895	6.261	7.462	5.414	7.340
0.6	3.602	2.206	2.747	4.978	4.585	5.911	6.286	6.221	4.169	6.072
0.7	3.520	2.257	2.506	4.718	4.348	5.933	6.317	4.965	2.889	2.879
0.8	3.434	2.311	2.255	4.481	4.104	5.960	6.354	3.692	1.602	3.485
0.9	3.344	2.371	1.994	4.236	3.852	5.994	6.398	2.402	0.297	2.163
1.0	3.250	2.435	1.723	3.982	3.591	6.035	6.450	1.093	-1.029	0.821
1.1	3.151	2.506	1.438	3.719	3.319	6.084	6.511	-0.237	-2.375	-0.544
1.3	2.937	2.666	0.828	3.162	2.743	6.213	6.666	-2.965	-5.141	-3.349
1.5	2.697	2.860	0.150	2.554	2.114	6.391	6.875	-5.800	-8.018	-6.270
1.7	2.424	3.097	-0.612	1.883	1.418	6.636	7.157	-8.764	-11.030	-9.334
1.9	2.108	3.391	-1.483	1.133	0.638	6.972	7.537	-11.888	-14.211	-12.576
2.1	1.736	3.765	-2.498	0.279	-0.254	7.435	8.056	-15.218	-17.610	-16.049
2.3	1.287	4.250	-3.713	-0.717	-1.297	8.081	8.771	-18.821	-21.299	-19.829
2.5	0.724	4.902	-5.214	-1.916	-2.555	9.001	9.783	-22.805	-25.392	-24.039
2.7	-9.835	5.811	-7.152	-3.419	-4.141	10.354	11.262	-27.352	-30.086	-28.890
2.9	-1.027	7.156	-9.810	-5.422	-6.201	12.443	13.532	-32.801	-35.744	-34.773
3.1	-2.566	9.313	-13.794	-8.337	-9.358	15.919	17.293	-39.875	-43.143	-42.522
3.3	-5.248	13.267	-20.674	-13.236	-14.584	22.480	24.368	-50.440	-54.289	-54.291
3.5	-11.343	22.634	-36.215	-24.044	-26.150	38.365	41.455	-71.322	-76.519	-77.971

TABLE 2 $\lambda L = 2$

0	4.546	1.596	4.546	9.060	4.273	4.273	9.060	34.413	4.951	34.413
0.2	4.439	1.636	4.183	8.752	3.849	4.187	9.137	32.297	2.657	31.720
0.4	4.326	1.682	3.801	8.433	3.403	4.101	9.232	30.155	0.324	28.953
0.6	4.206	1.735	3.396	8.101	2.933	4.017	9.348	27.985	-2.052	26.101
0.8	4.077	1.796	2.967	7.756	2.434	3.936	9.491	25.784	-4.476	23.153
1.0	3.939	1.868	2.507	7.396	1.904	3.859	9.665	23.549	-6.955	20.094
1.2	3.790	1.951	2.014	7.019	1.336	3.789	9.879	21.277	-9.495	16.906
1.4	3.629	2.050	1.480	6.622	0.723	3.727	10.142	18.962	-12.105	13.565
1.6	3.451	2.168	0.897	6.201	0.057	3.678	10.468	16.602	-14.796	10.043
1.8	3.254	2.311	0.255	5.753	-0.673	3.646	10.873	14.188	-17.584	6.300
2.0	3.034	2.485	-0.461	5.272	-1.484	3,639	11.383	11.712	-20.487	2.284
2.2	2.783	2.702	-1.271	4.750	-2.396	3.666	12.031	9.165	-23.531	-2.078
2.4	2.494	2.976	-2.203	4.177	-3.440	3.743	12.868	6.532	-26.753	-6.889
2.6	2.151	3.331	-3.301	3.537	-4.663	3.893	13.970	3.793	-30.204	-12.301
2.8	1.734	3.802	-4.630	2.806	-6.134	4.155	15.456	0.919	-33.968	-18.553
3.0	1.208	4.450	-6.302	1.945	-7.972	4.591	17.530	-2.137	-38.175	-26.042
3.2	0.509	5.384	-8.513	0.884	-10.386	5.318	20.555	-5.457	-43.062	-35.472
3.4	-0.487	6.826	-11.658	-0.515	-13.797	6.574	25.266	-9.165	-49.091	-48.250
3.6	-2.075	9.290	-16.655	-2.570	-19.178	8.913	33.380	-13.708	-57.337	-67.650
3.8	-5.129	14.327	-26.255	-6.235	-29.454	14.013	50.066	-20.048	-71.005	-103.48
4.0	-14.046	29.718	-54.250	-16.275	-59.275	30.305	101.260	-33.447	-106.13	-204.91
4.2	4883.2	-8646.2	15314.	5292.0	16265.	-9374.8	-28819.2	5783.6	17746.	54566.
4.4	20.508	-32.419	54.174	20.357	55.704	-37.953	-106.111	1.676	14.336	178.11

TABLE 3 $\lambda L = 3$

0	6.031	0.677	6.031	17.959	0.510	0.510	17.959	108.261	-9.109	108.261
0.25	5.961	0.665	5.682	17.755	0.049	0.260	18.160	105.978	-11.084	103.344
0.50	5.888	0.653	5.317	17.550	-0.431	0.005	18.385	103.669	-13.036	98.211
0.75	5.813	0.641	4.936	17.342	-0.931	-0.289	18.638	101.333	-14.959	92.834
1.00	5.734	0.629	4.536	17.133	-1.455	-0.593	18.925	98.967	-16.846	87.179
1.25	5.623	0.617	4.115	16.922	-2.006	-0919	19.253	96.566	-18.691	81.204
1.50	5.568	0.605	3.671	16.710	-2.586	-1.273	19.628	94.126	-20.481	74.860
1.75	5.479	0.594	3.198	16.497	-3.200	-1.658	20.062	91.641	-22.206	68.084
2.00	5.386	0.583	2.695	16.283	-3.854	-2.078	20.566	89.106	-23.847	60.800
2.25	5.287	0.574	2.154	16.069	-4.553	-2.542	21.159	86.509	-25.382	52.906
2.50	5.183	0.567	1.569	15.857	-5.306	-3.057	21.863	83.842	-26.786	44.277
2.75	5.073	0.564	0.932	15.647	-6.123	-3.635	22.708	81.089	-28.006	34.743
3.00	4.955	0.565	0.230	15.442	-7.019	-4.290	23.736	78.231	-29.000	24.076
3.25	4.827	0.575	-0.552	15.246	-8.012	-5.044	25.008	75.241	-29.683	11.962
3.50	4.689	0.596	-1.437	15.063	-9.130	-5.924	26.610	72.081	-29.944	-2.051
3.75	4.537	0.634	-2.458	14.902	-10.411	-6.974	28.675	68.698	-29.606	-18.632
4.00	4.368	0.700	-3.666	14.775	-11.916	-8.259	31.413	65.004	-28.395	-38.820
4.25	4.174	0.811	-5.144	14.707	-13.739	-9.886	35.175	60.861	-25.850	-64.331
4.50	3.947	1.000	-7.037	14.740	-16.052	-12.045	40.599	56.023	-21.135	-98.218
4.75	3.667	1.333	-9.631	14.961	-19.186	-15.108	48.955	50.005	-12.584	-146.512
5.00	3.291	1.966	-13.575	15.575	-23.894	-19.924	63.193	41.699	3.666	-223.067
5.25	2.699	3.365	-20.769	17.192	-32.373	-28.975	92.017	27.871	39.014	-368.503
5.50	1.326	7.768	-40.212	22.744	-55.029	-54.039	176.965	-6.843	147.839	-775.966
5.75	-36.823	159.233	-648.624	220.767	-758.383	-849.898	2985.955	-1047.078	3822.566	-13833.5
6.00	5.947	-13.411	40.010	-5.386	37.233	51.794	-206.406	127.210	358.700	972.160

TABLE 4 $\lambda L = 4$

0	7.987	-0.078	7.987	31.896	-1.757	-1.757	31.896	255.925	-13.142	255.925
0.5	7.912	-0.080	7.442	31.610	-2.170	-2.099	23.262	251.148	-13.821	240.758
1.0	7.835	-0.139	6.866	31.314	-2.575	-2.469	32.672	246.224	-14.149	224.537
1.5	7.754	-0.205	6.252	31.006	-2.969	-2.872	33.138	241.128	-14.060	207.078
2.0	7.670	-0.280	5.595	30.683	-3.347	-3.316	33.675	235.825	-13.465	188.148
2.5	7.581	-0.365	4.887	30.341	-3.701	-3.810	34.301	230.269	-12.251	167.445
3.0	7.486	-0.464	4.117	29.976	-4.021	-4.369	35.044	224.402	-10.266	144.571
3.5	7.385	-0.581	3.270	29.581	-4.291	-5.012	35.947	218.140	-7.302	118.986
4.0	7.276	-0.720	2.326	29.144	-4.489	-5.769	37.069	211.367	-3.069	89.937
4.5	7.156	-0.890	1.257	28.650	-4.579	-6.684	38.508	203.913	2.903	56.336
5..	7.022	-1.102	0.018	28.074	-4.503	-7.830	40.420	195.515	11.138	16.537
5.5	6.867	-1.378	-1.461	27.371	-4.164	-9.331	43.083	185.748	22.810	-32.091
6.0	6.680	-1.751	-3.297	26.464	-3.380	-11.416	47.012	173.867	39.703	-94.080
6.5	6.440	-2.289	-5.718	25.194	-1.781	-14.554	53.284	158.429	65.336	-178.074
7.0	6.099	-3.145	-9.222	23.191	1.503	-19.864	64.495	136.176	107.588	-303.251
7.5	5.516	-4.762	-15.221	19.348	9.088	-30.720	88.576	97.726	189.000	-523.954
8.0	4.029	-9.207	-30.207	8.355	33.586	-63.526	164.397	-0.803	414.451	-1091.15
8.5	-38.884	-142.482	-446.057	-338.300	857.156	-1139.434	2723.048	-2833.099	7153.89	-17181.6
9.0	9.792	8.925	22.605	58.353	-89.453	96.632	-225.008	375.164	-493.52	990.239
9.5	7.896	2.911	0.508	46.090	-63.040	63.655	-155.270	242.902	-176.03	164.685

TABLE 5 $\lambda L = 5$

0	9.966	-0.164	9.966	49.646	-1.277	-1.277	49.646	498.260	-4.511	498.260
0.5	9.917	-0.186	9.522	49.402	-1.304	-1.335	49.934	493.400	-3.514	478.078
1.0	9.866	-0.209	9.061	49.152	-1.306	-1.387	50.244	488.452	-2.296	456.999
1.5	9.814	-0.233	8.579	48.894	-1.282	-1.432	50.578	483.412	-0.835	434.923

2.0	9.761	-0.259	8.075	48.628	-1.227	-1.470	50.941	478.271	0.892	411.734
2.5	9.705	-0.286	7.545	48.353	-1.135	-1.498	51.338	473.019	2.913	387.294
3.0	9.648	-0.316	6.986	48.067	-1.002	-1.516	51.778	467.646	5.260	361.435
3.5	9.588	-0.347	6.394	47.770	-0.820	-1.520	52.270	462.139	7.973	333.956
4.0	9.524	-0.382	5.764	47.459	-0.580	-1.510	52.826	456.482	11.095	304.606
4.5	9.459	-0.420	5.090	47.133	-0.271	-1.482	53.464	450.660	14.682	273.075
5.0	9.389	-0.462	4.362	46.789	0.122	-1.433	54.207	444.649	18.799	238.966
5.5	9.314	-0.510	3.572	46.425	0.617	-1.360	55.088	438.425	23.530	201.765
6.0	9.235	-0.566	2.703	46.036	1.241	-1.258	56.155	431.956	28.977	160.791
6.5	9.148	-0.631	1.737	45.617	2.026	-1.120	57.477	425.204	35.273	115.115
7.0	9.052	-0.709	0.647	45.162	3.022	-0.943	59.162	418.120	42.593	63.432
7.5	8.946	-0.808	-0.609	44.662	4.299	-0.717	61.380	410.638	51.173	3.828
8.0	8.823	-0.937	-2.093	44.104	5.968	-0.439	64.417	402.675	61.348	-66.635
8.5	8.679	-1.116	-3.907	43.470	8.206	-0.106	68.780	394.113	73.613	-152.779
9.0	8.500	-1.379	-6.235	42.729	11.335	0.271	75.454	384.786	88.761	-263.200
9.5	8.262	-1.806	-9.447	41.828	15.996	0.638	86.572	374.442	108.192	-415.191
10.0	7.909	-2.600	-14.439	40.657	23.732	0.790	107.641	362.659	134.859	-650.414
10.5	7.252	-4.468	-24.161	38.900	39.605	-0.155	157.782	348.565	177.275	-1105.83
11.0	5.026	-12.370	-58.552	34.651	97.734	-9.149	370.881	328.657	287.408	-2705.96
11.5	15.691	31.268	110.703	48.761	-192.081	57.796	-806.335	332.680	-105.609	5129.63
12.0	9.800	9.701	18.909	38.240	-35.395	32.186	-223.739	294.368	173.531	863.267
12.5	8.586	7.886	1.517	33.271	-5.288	39.478	-172.904	252.680	297.185	37.706
13.0	7.428	9.335	-13.518	25.175	22.379	62.613	-207.043	176.410	496.599	-697.173

TABLE 6

Exact modified stability functions for p = O

λL	$s_1 = s_2$	sc	$Q_1 = Q_2$	$(qQ)_1 = (qQ)_2$	$T_1 = T_2$	Tt
1	4.038	1.972	6.208	5.877	13.480	11.491
2	4.550	1.600	9.073	4.280	34.438	4.962
3	6.039	0.678	18.007	0.507	108.408	-9.146
4	8.006	-0.030	32.049	-1.776	256.539	-13.246
5	10.003	-0.167	50.017	-1.293	500.104	-4.551

Modified stability functions by the Finite Difference Method for p = O

λL	$s_1 = s_2$	sc	$Q_1 = Q_2$	$(qQ)_1 = (qQ)_2$	$T_1 = T_2$	Tt
1	4.034	1.968	6.201	5.870	13.476	11.476
2	4.546	1.596	9.060	4.274	34.413	4.951
3	6.031	0.677	17.959	0.510	108.262	-9.109
4	7.987	-0.028	31.896	-1.757	255.925	-13.142
5	9.966	-0.164	49.646	-1.277	498.260	-4.511

APPENDIX II: Natations

$$A_i(x) = (1/\pi) \int_0^{\infty} \cos(\tfrac{1}{3}t^3 + xt)\, dt$$ Airy integral function

$A_n = (\pi^2 p / N^3)(n - 0.5) - 4$

$$B_i(x) = (1/\pi) \int_0^{\infty} \left(\sin(\tfrac{1}{3}t^3 + xt) + e^{(-\frac{1}{3}t^3 + xt)}\right) dt$$ Airy integral function

$B_n = 6 - (2n\pi^2 p / N^3) + 4(\lambda L / N)^4$

c = carry-over factor

$C_n = (\pi^2 p / N^3)(n + 0.5) - 4$

D_1, D_2, D_3, constants of integration

E = Young's modulus of beam-column

$$G_i(x) = (1/\pi) \int_0^{\infty} \sin(\tfrac{1}{3}t^3 + xt)\, dt$$ Airy integral function

$h = L/N$

I = moment of interia

k = modulus of subgrade reaction X width of beam-column

K = determinant of stiffness matrix

L = length

M = bending moment

N = number of increments

P = axial load

$P_e = \pi^2 EI / L^2$ Euler load

Q = sway moment factor

qQ = sway moment carry-over factor

s = stiffness factor

sc = moment carry-over factor

T = shear factor

tT = shear carry-over factor

y_1 = deflection of end 1 perpendicular to member 12

z = deflection at distance x along the strut

V = end shear force

$w = (P/EIL)^{\frac{1}{4}}$

θ = rotation

$p = P/P_e$ for strut carrying constant axial load

$\quad = P/2P_e$ for strut carrying axial load which is linearly varying

$\lambda = (k/4EI)^{\frac{1}{4}}$

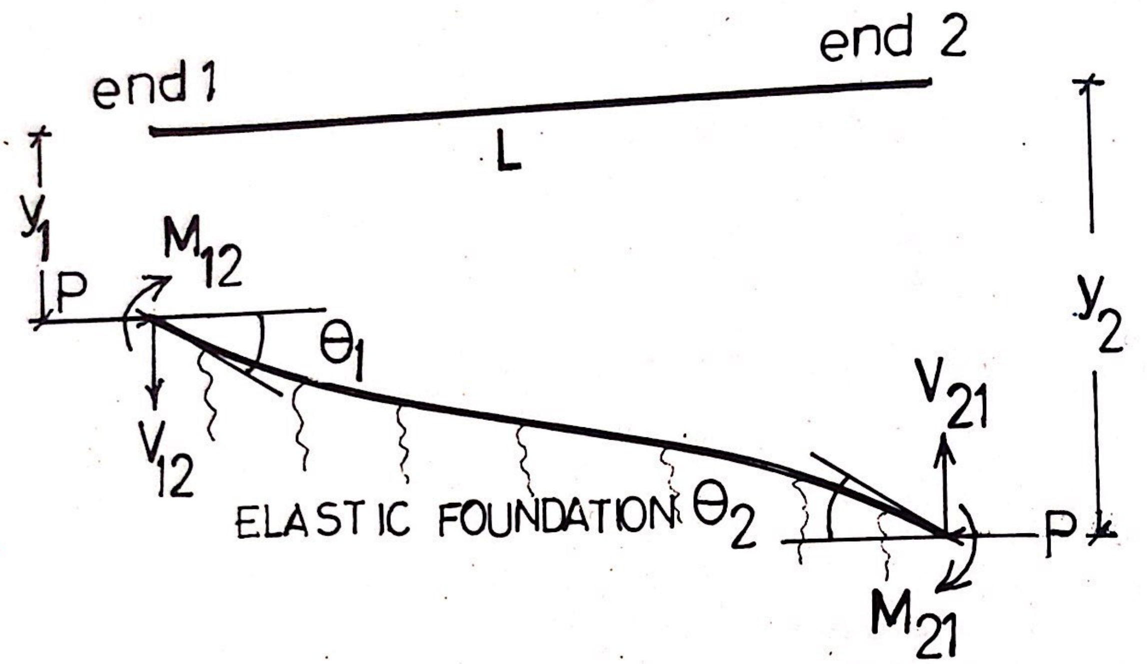

end 1
end 2
L
y_1
M_{12}
P
θ_1
V_{12}
V_{21}
y_2
ELASTIC FOUNDATION
θ_2
P
M_{21}

FIG. 1

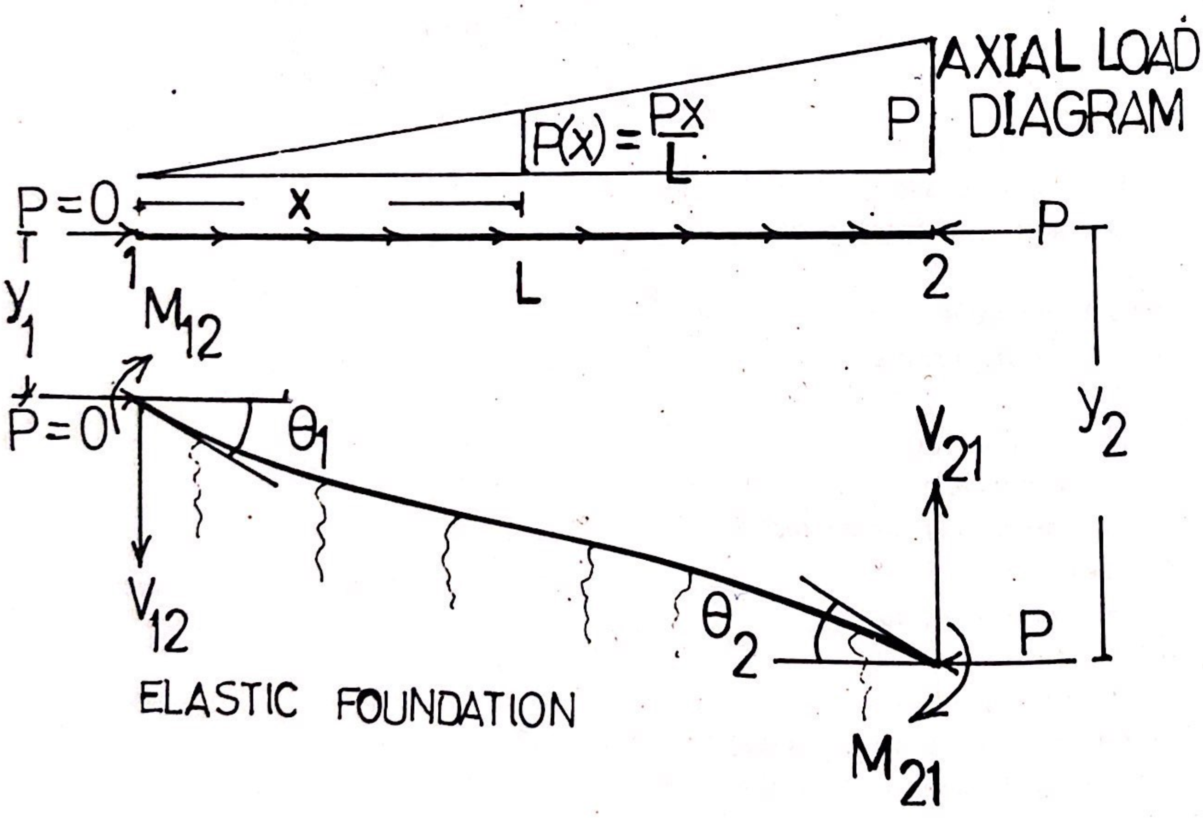

AXIAL LOAD
DIAGRAM
$P(x) = \frac{Px}{L}$
P
P=0
x
P
1
L
2
y_1
M_{12}
P=0
θ_1
V_{21}
y_2
V_{12}
θ_2
P
ELASTIC FOUNDATION
M_{21}

FIG. 2

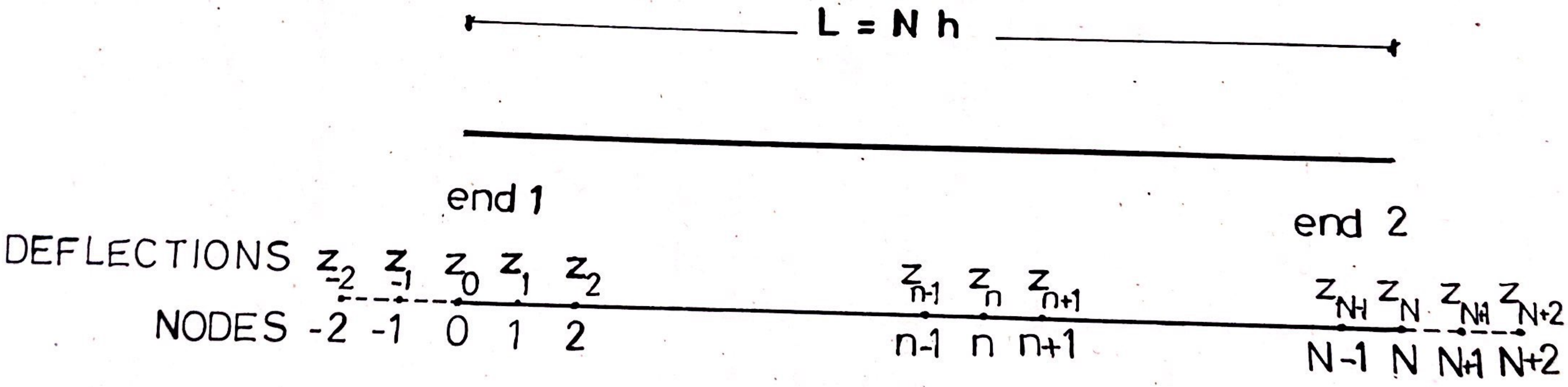
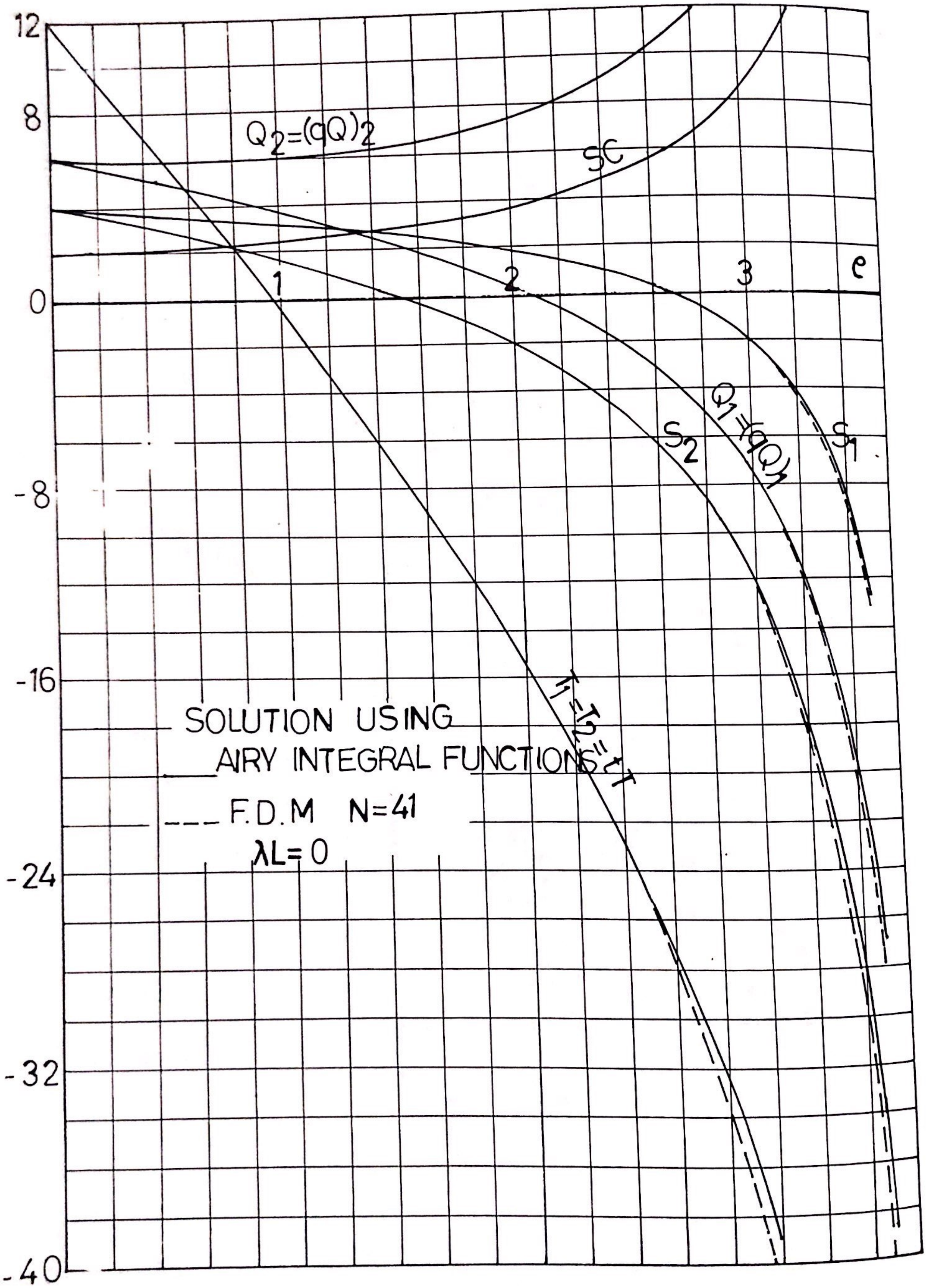


FIG. 3



12
8
Q2=(qQ)2
SC
e
1
2
3
0
Q1=(qQ)1
S2
S1
-8
-16
SOLUTION USING
AIRY INTEGRAL FUNCTIONS
T1=T2=tT
F.D.M N=41
λL=0
-24
-32
-40

FIG. 4

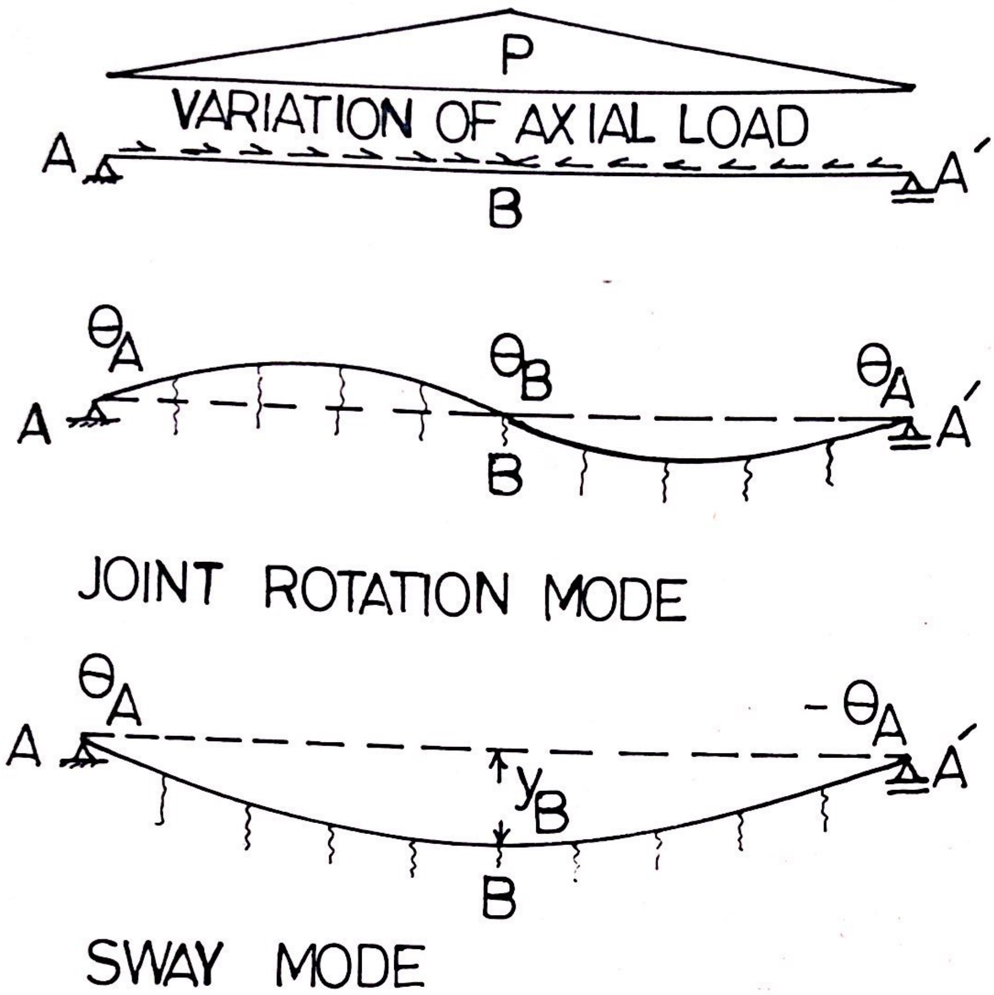

P
VARIATION OF AXIAL LOAD
A
A´
B
θA
θB
θA
A
B
A´
JOINT ROTATION MODE
θA
-θA
A
yB
A´
B
SWAY MODE

FIG. 5

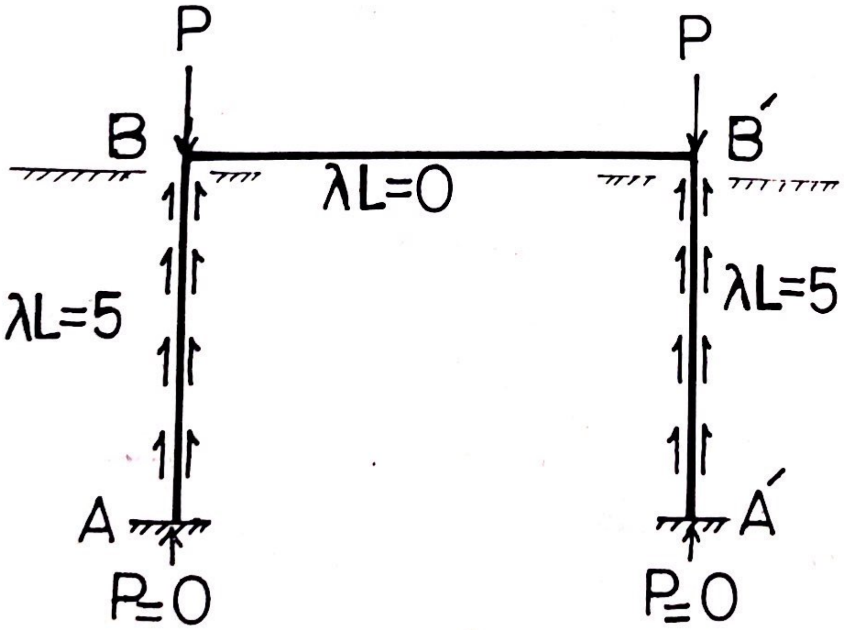

P
P
B
B´
λL=0
λL=5
λL=5
A
A´
P=0
P=0

FIG.6