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Elzaki Transform Method for Natural Frequency Analysis of Euler-Bernoulli Beams

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HIGHLIGHTS

- Natural frequencies of vibrating thin beams with different supports are found using the Elzaki transform method.
- The Elzaki transform converts the governing equation into an integral equation, simplifying the solution process.
- By considering boundary conditions, the transform space parameter facilitates a simplified solution approach.

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ABSTRACT

Free vibration analysis of thin beams under dynamic excitation is important in design to prevent resonance failures that occur when the excitation frequency coincides with the natural vibration frequency. In this paper, the Elzaki transform method (ETM) is used, for the first time, to solve the free vibration problem of thin beams. The beam is assumed to be homogeneous and prismatic, and the vibration is assumed to be harmonic. As a result, the field equation becomes a fourth-order homogeneous ordinary differential equation (ODE). The Elzaki transform method is chosen in this study due to its proven ability to solve ODEs, systems of ODEs, integro-differential equations, integral equations, and fractional differential equations. The Elzaki transformation simplifies the field equation into an algebraic equation in terms of the unknown deflection in the Elzaki space. By inverting the transformation, the general solution for the deflection is obtained in the physical domain, considering the initial conditions. The enforcement of boundary conditions for each case of end supports is utilized to determine the eigenequations, which are then solved for their roots using Symbolic Algebra methods. The eigenvalues are used to determine the exact natural frequencies of flexural vibration for each considered classical boundary condition. The eigenequations obtained are exact and identical to the ones previously derived by other scholars.

1. Introduction

Several theories have been developed to describe the behaviors of beams under flexural, vibrational and stability conditions. Some of the theories are Euler-Bernoulli beam theory (EBT), (for depth span ratios less than 0.05). Mindlin beam theory (MBT), Timoshenko beam theory (TBT), (for depth span ratios greater than 0.05) shear deformation beam theories proposed by Levinson, Reddy, and others. The analysis of the transverse vibrations of beams is important in the investigation of beams subjected to time dependent loadings in order to find out the natural frequencies of vibration associated with resonance failure. EBT does not account for shear deformations and rotary inertia in developing the governing equations. Consequently, EBT is strictly applied to thin beams and is the classical thin beam theory [1– 6]. MBT, TBT, Reddy beam theory (RBT), Levinson beam theory (LBT), and other shear deformable beam theories take account of the transverse shear deformations and rotary inertia in formulation of their field equations, and are thus utilized to describe moderately thick and thick beams [7 –11].

The field equation for beam vibration have been formulated using two basic methods. They are called equilibrium and variational methods. Equilibrium techniques apply D'Alembert's principle of dynamic equilibrium to express the field equation as a differential equation. Variational techniques apply the well known calculus of variations and energy minimization principles to express the problem using integral formulations. The field equations are thus solvable using classical and numerical methods for solving differential and integral equations.

Literature shows the beam vibration problems have been solved using the following solution methods (a) solution of variables (b) eigenfunction expansion (c) differential transform (d) Adomian decomposition (e) Laplace transformation (f)

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variational iteration (g) homotopy perturbation (h) finite difference (i) improved finite difference (j) finite element (k) point interpolation and (l) radial point interpolation.

To the author's knowledge, the Elzaki transformation methodology has not been previously applied for solutions to transverse natural vibrations analysis of thin beams within EBT.

Hurty and Rubinstein [12], have investigated the dynamics of structures using classical mathematical methods. The research by Hurty and Rubinstein [12], considered the applications of rigorous mathematical methods for solving partial differential equations to the solutions of structural dynamics problems of beams, and plates. They found analytical solutions for simple cases of homogeneous prismatic beams with classical boundary conditions. They found that analytical solutions could not be found as the dynamic beam problems became more complex with non-prismatic cross-sections and heterogeneous materials. Chun [13], studied the natural dynamics of thin beams with non-classical boundaries, such as sliding supports, yielding support etc. They found the problems involved transcendental equations which were difficult to solve. Kanbor and Tufik [14], have applied the Point Interpolation Methodology (PIM) and Radial Point Interpolation Methodology (RPIM) to the vibrating thin beam problem. Kanbor and Tufik [14], found that the PIM and RPIM reduce the vibrating thin beam ODEs to a system of algebraic equations which were solvable using matrix algebra methods. They however found that a large number of grid/or interpolation points were needed for accurate solutions. They validated their work with the illustrative problem of vibrating prismatic cantilever beams. Wang and Lin [15], and Kim and Kim [16], have used the method of orthogonal expansion in Fourier series to solve the vibrating thin beam problem under different boundary conditions. The research by Wang and Lin [15], and Kim and Kin [16], found that the method of orthogonal expansion in Fourier series effectively exploited the orthogonality properties of Fourier series expansions of the sought-for variables, thus simplifying the solution process. They found that exact solutions for the eigenvalues and eigenfunctions were obtained for problems of homogeneous beams with Dirichlet boundary conditions. Li [17], has solved the thin beam vibration problem.

The authors in [18], have applied Galerkin methodology to obtain solutions to vibrating non-prismatic thin beam problems. Achawakorn and Jearsiripongkul [18], research on the Galerkin variational solutions of the vibrating non-prismatic Euler-Bernoulli beam found that exact solutions were obtained when shape functions that satisfied both the displacement and force boundary conditions were employed in the Galerkin variational integral. Liu and Gurran [19], have applied He's Variational Iteration Method (VIM) to obtain solutions to the vibrating thin beam problem. Liu and Gurran [19], work on the application of the He's

VIM to the dynamic EB beam problem further demonstrated the effectiveness of the VIM by yielding very accurate solutions to the eigenvalue problem. Agboola et al. [20], have also applied the VIM to the free flexural vibration problem of prismatic cantilever beams and obtained accurate solutions. Sakman and Mutlu [21], have applied the method of perturbation to the free vibration problem of thin beams.

Hong et al. [22], used the method of separation of variables to obtain closed form solutions for the eigenvalues and eigenfrequencies of transverse vibration of clamped-pinned-free beam with mass at the free end. Their obtained frequencies were verified by using them to obtain frequencies for first five modes of vibration for three special cases; namely:

- Cantilever beam without attached mass.
- Cantilever beam with attached mass.
- Fixed-pinned-free beam with no attached mass.

Authors in [23], utilized the Differential Transform Methodology (DTM) to perform natural vibrations analysis of thin beams for a variety of boundaries; and developed results for natural frequencies that agreed with available results in the literature. Gritsenko et al. [24], proposed a new approach for finding analytical solutions for transverse vibration problems of cantilever beams subjected to the continuous, spatial distribution of load.

Coskun et al. [25], formulated the transverse vibration problems of thin beams using Adomian Decomposition Method (ADM), Variational Iteration Method (VIM) and Homotopy Perturbation Method (HPM). They applied their formulation to determine the natural frequencies for prismatic and non-prismatic thin beams for varieties of boundaries and developed satisfactory eigenfrequencies solutions that agreed with exact eigenfrequencies. Adair and Jaeger [26], used the Adomian modified decomposition method (AMDM) to solve the transverse vibration problems of non-prismatic cantilever Timoshenko beams subjected to a transversely and axially eccentric tip mass. Torabi et al [27], utilized the Variational Iteration Methodology (VIM) to study linear and non-linear transverse vibration problems of thin beams. Chalah-Rezgni et al. [28], used the finite element method to perform the transverse vibration analysis of prismatic thin beams under various boundary conditions. Flaieh et al [29], investigated the analysis of clamped-free beams assuming various geometric parameters and material properties.

Al-Raheimy [30], using the separation of variables method presented analytical solutions for natural transverse vibrations of uniform beams with simply supported boundaries under static axial force. The effect of tensile and compressive axial force on the natural frequencies and mode shape functions were studied. It was found that tensile axial force increases the natural frequencies while increased compressive axial force results in reduced natural frequency.

Rama [31], applied the Elzaki transform method to the analysis of undamped, damped and forced vibrations of mechanical systems. Rama [31] transformed the governing single degree of freedom (SDOF) equation of motion using the ETM to an algebraic equation in the Elzaki-transformed space. The researcher in [31], was able to derive analytical solutions to the resulting problem for undamped, damped, and forced vibrations of mechanical systems considered. Ramachandruni and Namala [32], have applied Elzaki transform method to obtain analytical solutions to some problems in engineering mechanics.

Elzaki et al. [33], have used the Elzaki transform method to solve differential equations governing mechanics, electrical circuits, and beam flexure problems. Lim and Chan [34], presented a novel static approach for natural vibration solutions of beams subject to a variety of boundary conditions. In their new static method, the field equation for beams under sinusoidal loads and supported on elastic foundations are solved and the solutions are utilized to obtain the beam-free vibrations eigenfrequencies.

Ike [35], has applied Sumudu transformation methodology (STM) to solve the eigenvalue problem of natural transverse vibrations of thin beams and obtained exact solutions for the eigenfrequencies and eigenfunctions for various classical boundary conditions considered.

The review of the literature reveals that the ETM has not been applied so far for studying eigenvalue problems of vibrating thin beams and that is the focus of this research.

In this work, the Elzaki transformation methodology (ETM) is utilized in a novel way to obtain solutions to the natural flexural vibration frequencies of Euler-Bernoulli beams under different boundary conditions. The ETM is utilized due to:

a) Its advantage over other techniques is that it gives the complete solution to the governing field equation.
b) It presents the problem in terms of the initial conditions which are thus fully considered in solving the problem.
c) It offers a reduction in computational rigor since the problem is simplified to an algebraic one which is easier to solve.
d) It is systematic and can be generalized.
e) The inversion of the Elzaki transform is simple to do since a catalog of inverse transforms exists which can be used.

The disadvantage of the ETM is that in some problems, the inversion of the transformed algebraic expression in the Elzaki space would require the solution of complex problems of contour integration.

2. Theory

The governing equation is [6,12]:

$$EI\frac{\partial^4 v(x,t)}{\partial x^4}+m\frac{\partial^2 v(x,t)}{\partial t^2}=0 \tag{1}$$

where $v(x, t)$ is the transverse vibration, x is the longitudinal coordinate, t is the time, E is Young's modulus, I is the moment of inertia, and m is the beam mass per unit length.

Assuming sinusoidal excitations, the response is expectedly sinusoidal, and hence let

$$v(x,t)=V(x)\exp(i\omega_n t) \tag{2}$$

where ω_n is the natural vibration frequency $V(x)$ is the vibration mode shape function, i is the imaginary number $i=\sqrt{-1}$

Then, Equation (1) becomes:

$$\left(EI\frac{d^4V(x)}{dx^4}-m\omega_n^2 V(x)\right)\exp(i\omega_n t)=0 \tag{3}$$

For non trivial solutions, $\exp(i\omega_n t)\neq 0,$ then the problem simplifies to the fourth order ordinary differential equation (ODE):

$$\frac{d^4V(x)}{dx^4}-\frac{m\omega_n^2}{EI}V(x)=0 \tag{4}$$

$$\text{or, } \frac{d^4V(x)}{dx^4}-\lambda_n^4 V(x)=0 \tag{5}$$

$$\text{where } \lambda_n^4=\frac{m\omega_n^2}{EI} \tag{6}$$

λ_n^4 is the natural frequency parameter.

3. Methodology

The Elzaki transform of the field equation is [31]

$$u\int_0^{\infty}\left(\frac{d^4V(x)}{dx^4}-\lambda_n^4 V(x)\right)e^{-x/u}dx=0 \tag{7}$$

where u is Elzaki transformation parameter.

By linearity properties of the Elzaki transform, Equation (7) is simplified as:

$$u\int_0^\infty \frac{d^4V(x)}{dx^4}e^{-x/u}du-\lambda_n^4u\int_0^\infty V(x)e^{-x/u}dx=0 \tag{8}$$

The integral transform is evaluated as: the algebraic equation:

$$\frac{V(u)}{u^4}-\frac{V(0)}{u^2}-\frac{V'(0)}{u}-V''(0)-uV'''(0)-\lambda_n^4V(u)=0 \tag{9}$$

$$\text{where } V(u)=u\int_0^\infty V(x)e^{-x/u}dx \tag{10}$$

V(*u*) is the Elzaki transformation of *V*(*x*)Simplifying,

$$\frac{V(u)}{u^4}-\lambda_n^4V(u)=\frac{V(0)}{u^2}+\frac{V'(0)}{u}+V''(0)+uV'''(0) \tag{11}$$

Solving for *V*(*u*) gives:

$$V(u)=V(0)\left(\frac{u^2}{1-\lambda_n^4u^4}\right)+V'(0)\left(\frac{u^3}{1-\lambda_n^4u^4}\right)+V''(0)\left(\frac{u^4}{1-\lambda_n^4u^4}\right)+V'''(0)\left(\frac{u^5}{1-\lambda_n^4u^4}\right) \tag{12}$$

By inversion,

$$V(x)=T^{-1}V(u) \tag{13}$$

where T^{-1} is the inverse Elzaki transform of $V(u)T^{-1}$ is a linear operator, and hence *V*(*x*) becomes:

$$V(x)=V(0)T^{-1}\left(\frac{u^2}{1-\lambda_n^4u^4}\right)+V'(0)T^{-1}\left(\frac{u^3}{1-\lambda_n^4u^4}\right)+V''(0)T^{-1}\left(\frac{u^4}{1-\lambda_n^4u^4}\right)+V'''(0)T^{-1}\left(\frac{u^5}{1-\lambda_n^4u^4}\right) \tag{14}$$

Using partial fraction decomposition techniques and the catalogue of inverse Elzaki transforms, we have:

$$T^{-1}\frac{u^2}{1-\lambda_n^4u^4}=\frac{\cosh\lambda_nx+\cos\lambda_nx}{2} \tag{15}$$

$$T^{-1}\frac{u^3}{1-\lambda_n^4u^4}=\frac{\sinh\lambda_nx+\sin\lambda_nx}{2\lambda_n} \tag{16}$$

$$T^{-1}\frac{u^4}{1-\lambda_n^4u^4}=\frac{\cosh\lambda_nx-\cos\lambda_nx}{2\lambda_n^2} \tag{17}$$

$$T^{-1}\frac{u^5}{1-\lambda_n^4u^4}=\frac{\sinh\lambda_nx-\sin\lambda_nx}{2\lambda_n^3} \tag{18}$$

Hence,

$$V(x)=V(0)\left(\frac{\cosh\lambda_nx+\cos\lambda_nx}{2}\right)+V'(0)\left(\frac{\sinh\lambda_nx+\sin\lambda_nx}{2\lambda_n}\right)+$$

$$V''(0)\left(\frac{\cosh\lambda_nx-\cos\lambda_nx}{2\lambda_n^2}\right)+V'''(0)\left(\frac{\sinh\lambda_nx-\sin\lambda_nx}{2\lambda_n^3}\right) \tag{19}$$

Differentiating,

$$V'(x)=\theta(x)=V(0)\lambda_n\left(\frac{\sinh\lambda_n x-\sin\lambda_n x}{2}\right)+V'(0)\left(\frac{\cosh\lambda_n x+\cos\lambda_n x}{2}\right)+$$
$$V''(0)\left(\frac{\sinh\lambda_n x+\sin\lambda_n x}{2\lambda_n}\right)+V'''(0)\left(\frac{\cosh\lambda_n x-\cos\lambda_n x}{2\lambda_n^2}\right) \quad (20)$$

$$V''(x)=V(0)\lambda_n^2\left(\frac{\cosh\lambda_n x-\cos\lambda_n x}{2}\right)+V'(0)\lambda_n\left(\frac{\sinh\lambda_n x-\sin\lambda_n x}{2}\right)+$$
$$V''(0)\left(\frac{\cosh\lambda_n x+\cos\lambda_n x}{2}\right)+V'''(0)\left(\frac{\sinh\lambda_n x+\sin\lambda_n x}{2\lambda_n}\right) \quad (21)$$

$$V'''(x)=V(0)\lambda_n^3\left(\frac{\sinh\lambda_n x+\sin\lambda_n x}{2}\right)+V'(0)\lambda_n^2\left(\frac{\cosh\lambda_n x-\cos\lambda_n x}{2}\right)+$$
$$V''(0)\lambda_n\left(\frac{\sinh\lambda_n x-\sin\lambda_n x}{2}\right)+V'''(0)\left(\frac{\cosh\lambda_n x+\cos\lambda_n x}{2}\right) \quad (22)$$

4. Results

4.1 Simply Supported Ends (SS)

Thin beam with simple supports at the ends $x = 0$, and $x = l$, as shown in Figure 1 is considered.

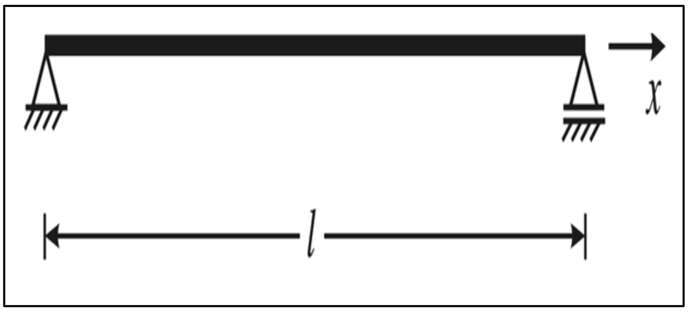

Figure 1: Simply supported thin beam

The boundary conditions are:

$$V(0)=V(l)=0 \quad (23a)$$

$$V''(0)=V''(l)=0 \quad (23b)$$

Application of Equations (23a) and (23b) give:

$$\begin{pmatrix}\left(\frac{\sinh\lambda_n l+\sin\lambda_n l}{2\lambda_n}\right) & \left(\frac{\sinh\lambda_n l-\sin\lambda_n l}{2\lambda_n^3}\right)\\ \lambda_n\left(\frac{\sinh\lambda_n l-\sin\lambda_n l}{2}\right) & \left(\frac{\sinh\lambda_n l+\sin\lambda_n l}{2\lambda_n}\right)\end{pmatrix}\begin{pmatrix}V'(0)\\ V'''(0)\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \quad (24)$$

The frequency equation is:

$$\begin{vmatrix} \left(\dfrac{\sinh\lambda_n l + \sin\lambda_n l}{2\lambda_n}\right) & \left(\dfrac{\sinh\lambda_n l - \sin\lambda_n l}{2\lambda_n^3}\right) \\ \lambda_n\left(\dfrac{\sinh\lambda_n l - \sin\lambda_n l}{2}\right) & \left(\dfrac{\sinh\lambda_n l + \sin\lambda_n l}{2\lambda_n}\right) \end{vmatrix} = 0 \quad (25)$$

Expansion gives the eigenequation:

$$\sin\lambda_n l = 0 \quad (26)$$

Solving,

$$\lambda_n = \frac{n\pi}{l} = \left(\frac{m\omega_n}{EI}\right)^{1/4} \quad (27)$$

Hence,

$$\omega_n = \left(\frac{n\pi}{l}\right)^2 \sqrt{\frac{EI}{m}} = (n\pi)^2 \sqrt{\frac{EI}{ml^2}} = \overline{\omega}_n \sqrt{\frac{EI}{ml^2}} \quad (28)$$

The results for the first four eigenfrequencies are tabulated in Table 1.

Table 1: Non-dimensional frequencies $\overline{\omega}_n$ for simply supported thin beams

Vibration mode	Avcar [6], Hurty and Rubinstein [12], Ike [35]	Present work
1	9.869604401	9.869604401
2	39.4784176	39.4784176
3	88.82643961	88.82643961
4	157.9136704	157.9136704

4.2 Thin Beam With Clamped Ends (CC)

A thin beam with clamped ends as shown in Figure 2 is considered.

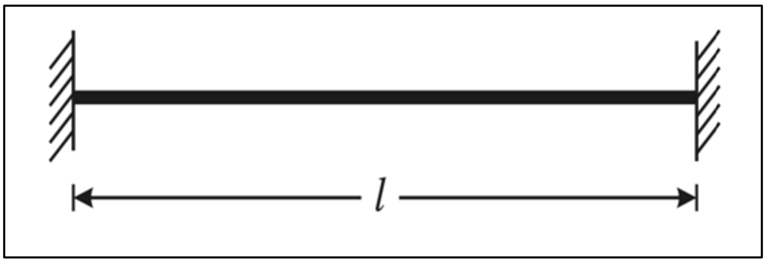

Figure 2: Thin beam with clamped ends

The boundary conditions are:

$$V(0) = V'(0) = 0 \quad (29a)$$

$$V(l) = V'(l) = 0 \quad (29b)$$

The enforcement of Equations (29a) and (29b) give:

$$\begin{pmatrix} \left(\dfrac{\cosh\lambda_n l - \cos\lambda_n l}{2\lambda_n^2}\right) & \left(\dfrac{\sinh\lambda_n l - \sin\lambda_n l}{2\lambda_n^3}\right) \\ \left(\dfrac{\sinh\lambda_n l + \sin\lambda_n l}{2\lambda_n}\right) & \left(\dfrac{\cosh\lambda_n l - \cos\lambda_n l}{2\lambda_n^2}\right) \end{pmatrix} \begin{pmatrix} V''(0) \\ V'''(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (30)$$

The characteristic frequency equation is obtained for nontrivial solutions as shown:

$$\begin{vmatrix} \left(\dfrac{\cosh\lambda_n l - \cos\lambda_n l}{2\lambda_n^2}\right) & \left(\dfrac{\sinh\lambda_n l - \sin\lambda_n l}{2\lambda_n^3}\right) \\ \left(\dfrac{\sinh\lambda_n l + \sin\lambda_n l}{2\lambda_n}\right) & \left(\dfrac{\cosh\lambda_n l - \cos\lambda_n l}{2\lambda_n^2}\right) \end{vmatrix} = 0 \quad (31)$$

Expansion gives the eigenequation:

$$1 - \cosh\lambda_n l \cos\lambda_n l = 0 \quad (32)$$

The frequency equation is a transcendental equation with an infinite number of eigenvalues (roots). The eigenvalues are extracted using Symbolic Algebra software, Mathematica software or other numerical analysis tools as:

$$\begin{aligned} \lambda_1 l &= 4.73014 \\ \lambda_2 l &= 7.85321 \\ \lambda_3 l &= 10.9956 \\ \lambda_4 l &= 14.1372 \\ \lambda_n l &= \frac{(2n+1)\pi}{2} \end{aligned} \quad (33)$$

The natural frequency for the *n*th vibratory mode is

$$\omega_n = \lambda_n^2\sqrt{\frac{EI}{m}} = \overline{\omega}_n\sqrt{\frac{EI}{ml^2}} \quad (34)$$

The results for the first four modes of vibration are shown in Table 2

Table 2: Dimensionless frequencies $\overline{\omega}_n$ of thin beam with fixed ends

Mode of vibration	Avcar [6], Hurty and Rubinstein [12], Ike [35]	Present work
1	22.37384601	22.37384601
2	61.67275024	61.67275024
3	120.9032194	120.9032194
4	199.8604238	199.8604238

4.3 Results for Thin Beam With Clamped-Free Ends (CFree)

A thin beam of length *l*, with clamped free end, as shown in Figure 3, is considered.

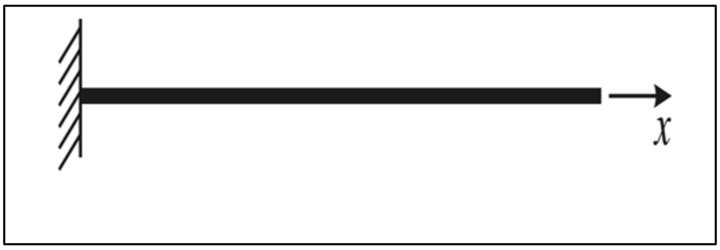

Figure 3: Thin beam clamped at $x = 0$ and free at $x = l$

The boundary conditions are:

$$V(0) = \theta(0) = V'(0) = 0 \quad (35a)$$

$$V''(l) = V'''(l) = 0 \quad (35b)$$

Application of Equations (35a) and (35b) give the characteristic frequency equation as:

$$\begin{vmatrix} \left(\dfrac{\cosh\lambda_n l + \cos\lambda_n l}{2}\right) & \left(\dfrac{\sinh\lambda_n l + \sin\lambda_n l}{2\lambda_n}\right) \\ \lambda_n\left(\dfrac{\sinh\lambda_n l - \sin\lambda_n l}{2}\right) & \left(\dfrac{\cosh\lambda_n l + \cos\lambda_n l}{2}\right) \end{vmatrix} = 0 \quad (36)$$

Expanding gives after simplifying, the eigenequation:

$$1 + \cosh\lambda_n l \cos\lambda_n l = 0 \quad (37)$$

The solution using mathematical software tools gives the first ten eigenvalues as:

$$\begin{aligned} \lambda_1 l &= 1.875104069 \\ \lambda_2 l &= 4.694091133 \\ \lambda_3 l &= 7.854757438 \\ \lambda_4 l &= 10.99554073 \\ \lambda_5 l &= 14.13716831 \\ \lambda_6 l &= 17.27875953 \\ \lambda_7 l &= 20.42035225 \\ \lambda_8 l &= 23.5619449 \\ \lambda_9 l &= 26.703537556 \\ \lambda_{10} l &= 29.84513021 \end{aligned} \quad (38)$$

for $\boldsymbol{n \geq 3, \lambda_n l \frac{(2n-1)\pi}{2}}$

$$\boldsymbol{\omega_n = \lambda_n^2\sqrt{\frac{EI}{m}} = \varpi_n\sqrt{\frac{EI}{ml^2}}}$$

The dimensionless eigenfrequencies $\overline{\omega}_n$ of cantilever beams are presented in Table 3 for the first four modes.

Table 3: Dimensionless natural frequencies of thin beam clamped at one end and free at the other end

Mode of vibration	Avcar [6]	Ike [35]	Present work
1	3.5160	3.516015011	3.516015011
2	22.0345	22.03449032	22.03449032
3	61.6921	61.69720753	61.69720753
4	120.9019	120.9019153	120.9019153

4.4 Results for Thin Beam Fixed at x = 0, Simply Supported at x = l (CS)

The clamped-simply supported (CS) beam shown in Figure 4 is considered.

Figure 4: Thin beam with clamped-simply supported ends

The boundary conditions are given by:

$$V(0) = \theta(0) = V'(0) = 0 \quad (39a)$$

$$V(l) = V''(l) = 0 \quad (39b)$$

Equations (39a) and (39b) are used to obtain the following.

$$\begin{vmatrix} \dfrac{\cosh\lambda_n l - \cos\lambda_n l}{2\lambda_n^2} & \dfrac{\sinh\lambda_n l - \sin\lambda_n l}{2\lambda_n^3} \\ \dfrac{\cosh\lambda_n l + \cos\lambda_n l}{2} & \dfrac{\sinh\lambda_n l + \sin\lambda_n l}{2\lambda_n} \end{vmatrix} = 0 \quad (40)$$

Expanding and simplifying yields the eigenfrequency equation as the transcendental equation – Equation (41) – which is similar to equations derived by Avcar [6], and Ike [35]

$$\tan\lambda_n l = \tanh\lambda_n l \quad (41)$$

The first four eigenvalues of Equation (41) are:

$$\begin{aligned} \lambda_1 l &= \pm 3.92660231204792 \\ \lambda_2 l &= \pm 7.06858274562873 \\ \lambda_3 l &= \pm 10.2101761228130 \\ \lambda_4 l &= \pm 13.3517687777541 \end{aligned} \quad (42)$$

Using Equation (34) ω_n is found from λ_n, $\omega_n = \lambda_n^2\sqrt{\dfrac{EI}{m}} = \bar{\omega}_n\sqrt{\dfrac{EI}{ml^2}}$

Identical results are found for thin beams with simple supports at $x = 0$ and fixed at $x = l$ (SC beams). The natural frequencies for the first four modes of vibration of thin CS and SC beams are shown in Table 4.

Table 4: Dimensionless natural frequencies of CS and SC thin beams

Vibration mode	Avcar [6]	Ike [35]	Present results
1	15.4182	15.41820572	15.41820572
2	49.9649	49.96486202	49.96486202
3	104.2477	104.2476964	104.2476964
4	178.2697	178.2697293	178.2697293

5. Discussion

In this paper, the ETM is applied for the first time to solve the fourth order partial differential equation for flexural vibrations of thin beams. The assumption of harmonic transverse vibrations and prismatic cross-section reduces the problem to an ordinary differential equation (ODE) of the fourth order presented in Equation (4). The Elzaki transformation of the governing fourth order ODE presented in Equation (7) simplifies the ODE to an algebraic problem – Equation (12) – in the Elzaki transform space. The general solution in the physical domain is found by inversion of the Elzaki transformed equation. Solutions are obtained for classical boundary conditions. Consideration of boundary conditions for simply supported beams reduced the problem to the homogeneous algebraic eigenvalue problem in Equation (24). The frequency equation obtained for simply supported beams is determined as Equation (25) which upon expansion gave the eigenequation in Equation (26). The solution of Equation (26) gives the eigenvalues given in Equation (27) and the exact natural frequencies which are given in Table 1. Table 1 shows exact agreement with previous works of Avcar [6], Hurty and Rubinstein [12] and Ike [35]. Similarly, for thin beam with clamped ends, the consideration of boundary conditions gives Equation (30). The resulting frequency equation is found as Equation (32) and is obtained from the expansion of the determinant in Equation (31). The eigenequation in Equation (32) is a transcendental equation with infinite number of eigenvalues (roots). The roots are obtained using Symbolic Algebra software, Mathematica software and other numerical analysis tools as Equation (33). The resulting natural frequencies of CC thin beams are shown in Table 2. Table 2 demonstrates the exact agreement of the present results for CC beams with previous works of Avcar [6], Ike [35] and Hurty and Rubinstein [12]. For thin beams with C-Free ends, the frequency equation is obtained as Equation (36). The eigenequation becomes the transcendental equation – Equation (37) – obtained upon expansion of Equation (36). Equation (37) has infinite number of eigenvalues and the first ten eigenvalues obtained using mathematical software are given in Equation (38). The natural frequencies of thin cantilever beams for the first four modes of vibration are shown in Table 3. Table 3 illustrates the agreement of the present results with the previous work of Avcar [6], and Ike [35]. The frequency equation obtained from the boundary conditions of beams with clamped-simply supported ends is shown in Equation (40). The determinant in Equation (40) is expanded to obtain the eigenequation as in Equation (41). The first ten eigenvalues of Equation (41) are presented as Equation (42) and the first four natural frequencies are shown in Table 4. Table 4 shows excellent agreement with results presented by Ike [35], and Avcar [6].

6. Conclusion

The ETM has been used in this paper for developing exact eigenfrequencies of free vibrations of thin beams within Euler-Bernoulli theory. The ETM has the merit that it does not require pre-selection of the basis functions since the eigenfunctions of vibrating thin beams are used as kernel functions in the resulting integral equation.

1) The ETM converts the fourth-order differential equation to an algebraic problem in the Elzaki transformed coordinates.
2) The algebraic equation in the transformed space is in terms of the initial conditions of the problem.
3) The general solution is obtained in the physical domain space by use of the inverse Elzaki transform techniques and partial fraction decomposition methods for the reduction of complicated algebraic expressions into simpler ones for which inverse Elzaki transforms exist.
4) The enforcement of the boundary conditions for the considered cases reduces the problem to eigenequations which are transcendental equations, solvable using Mathematica software, Symbolic software, and numerical and iterative techniques.
5) The eigenvalues of the eigenequations gave the exact solutions for the natural frequencies of thin beams for the various boundary conditions considered in this work.

Nomenclature/Notation/Symbols

EBT	Euler-Bernoulli beam theory
MBT	Mindlin beam theory
TBT	Timoshenko beam theory
RBT	Reddy beam theory
LBT	Levinson beam theory
ETM	Elzaki transform method
ODE	Ordinary differential equation
DTM	Differential transform methodology
VIM	Variational iteration method/methodology
ADM	Adomian decomposition method
AMDM	Adomian modified decomposition method
HPM	Homotopy perturbation method
STM	Sumudu transformation methodology
PIM	Point interpolation methodology
RPIM	Radial point interpolation methodology
EB	Euler-Bernoulli
x	Longitudinal coordinate of beam
t	Time
$v(x, t)$	Transverse displacement of beam
E	Young's modulus of beam material
I	Moment of inertia
m	Mass per unit length of beam
ω_n	Natural vibration frequency
$V(x)$	Vibration mode shape function
i	Imaginary number
λ_n^4	Natural frequency parameter
u	Elzaki transform parameter
∞	Infinity
$\int$	Integral
$\int(\ )\,dx$	Integration with respect to x
$V(u)$	Elzaki transform of $V(x)$
T^{-1}	Inverse Elzaki transform operator
T	Transform operator
$\overline{\omega}_n$	Non-dimensional frequencies
$\frac{\partial}{\partial x}$	partial differential operator (with respect to x)
$\frac{\partial}{\partial t}$	partial differential operator (with respect to t)
SS	Simply supported (at two ends, $x = 0$, $x = l$)
CC	Clamped (at two ends, $x = 0$, $x = l$)
CFree	Clamped-free (clamped at $x = 0$, free at $x = l$)
CS	Clamped-simply supported (clamped at $x = 0$, simply supported at $x = l$)
SC	Simply supported clamped (simply supported at $x = 0$, clamped at $x = l$)

Author contributions

Conceptualization, C. C. Ike and T. M. Elzaki; methodology, C. C. Ike and T. M. Elzaki; Software, C. C. Ike and T. M. Elzaki; validation, C. C. Ike and T. M. Elzaki; formal analysis, C. C. Ike and T. M. Elzaki; investigation, C. C. Ike and T. M. Elzaki; resources, C. C. Ike; data curation, C. C. Ike and T. M. Elzaki; writing—original draft preparation, C. C. Ike and T. M. Elzaki; writing—review and editing, C. C. Ike and T. M. Elzaki; visualization, C. C. Ike and T. M. Elzaki; supervision, C. C. Ike and T. M. Elzaki; project administration, C. C. Ike and T. M. Elzaki; funding acquisition, C. C. Ike; All authors have read and agreed to the final version of the manuscript.

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The data that support the findings of this study are available on request from the corresponding author.

Conflicts of interest

The authors declare that there is no conflict of interest.

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