Engineering and Technology Journal
Journal homepage: https://etj.uotechnology.edu.iq

An Analytical Solution to Buckling of Thick Beams Based on a Cubic Polynomial Shear Deformation Beam Theory

Charles Chinwuba Ike

Department of Civil Enginnering, Enugu State, University of Science And Technology - Nigeria, Agbani, Enugu State, Nigeria.
*Corresponding author Email: charles.ike@esut.edu.ng

HIGHLIGHTS

- Closed form solutions are derived for the buckling of thick beams under in-plane loading.
- The buckling peoblem is derived using cubic polynomial shear deformation theory.
- For thin beams ($h/l = 0.01$,), $N_{xx\ cr}$ has a negligible difference from the BEBT results.
- For $h/l = 0.25$ (thick beam), $N_{xx\ cr}$ obtained is 13.44% lower than the value predicted by BEBT.

ARTICLE INFO

Handling editor: Ayad M. M. Jubori

Keywords:

Cubic polynomial shear deformation beam; Buckling theory; Finite sine transform; Potential energy functional; Euler-Lagrange equation; Algebraic eigenvalue problem.

ABSTRACT

This paper presents analytical solutions for the buckling of thick beams. The Bernoulli-Euler beam theory (BEBT) overestimates their critical buckling load. This paper has derived a cubic polynomial shear deformation beam buckling theory (CPSDBBT) from first principles using the Euler-Lagrange differential equation (ELDE). It develops closed-form solutions to differential equations using the finite sine transform method. The formulation considers transverse shear deformation and satisfies the transverse shear stress-free boundary conditions. The governing equation is developed from the energy functional, $\prod$, by applying the ELDE. The domain equation is obtained as an ordinary differential equation (ODE). The finite sine transformation of the ODE transforms the thick beam, which is considered an algebraic eigenvalue problem. The solution gives the buckling load N_{xx} at any buckling mode n. The critical buckling load $N_{xx\ cr}$ occurs at the first buckling mode and is presented in depth ratios to span (h/l). It is found that and agrees with previous solutions using shear deformable theories. For $h/l = 0.10$, (a moderately thick beam), the $N_{xx\ cr}$ is 2.50% lower than the value predicted using BEBT, confirming the overestimation by BEBT. The $N_{xx\ cr}$ $h/l = 0.10$, agrees with previous solutions, implying the shear deformation has been adequately accounted for, and the BEBT overestimates the $N_{xx\ cr}$. The value of $N_{xx\ cr}$ found agrees with previous values in the literature.

1. Introduction

Beams are structures or structural members usually subjected to point or distributed transverse forces and moments that produce transverse deflections and rotations. They are commonly found in bridges, buildings, airplanes, and machines. They are classified based on the ratios of their depth, h to length (span) l as thin when and as moderately thick or thick when

Beams with circular cross-sections are classified as thin when their diameter ratio, d, to span, l, is less than 0.05 and as thick when the ratio of diameter to span is greater than 0.05.

Figures 1a and b show thick and thin beams with rectangular cross-sections based on their depth-to-span ratios. Figures 2a and b show thin and thick beams with circular cross-sections based on their diameter ratios, d, to span, l. Single-span beams can be subject to different kinds of supports at the ends. Common ways of supporting beams at the ends are illustrated in Figures 3(a-d) and are listed as simple supports (in Figure 3a), clamped fixed supports (in Figure 3b), clamped-simple supports (in Figure 3c), and clamped-free supports (in Figure 3d).

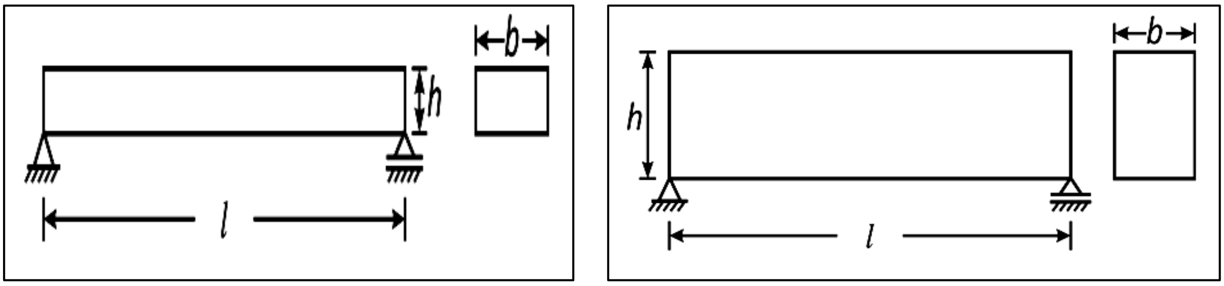
Several theories have been proposed to describe the structural behaviors of beams under bending, dynamic, and buckling forces. The well-known Bernoulli-Euler beam theory (BEBT) is called the classical thin beam theory (CTBT). The theory is based on the plane cross-section deformation hypothesis, which states that plane cross-sections originally orthogonal to the beams' neutral axis before deformation remain plane and orthogonal to the neutral axis after deformation. Consequently, shear strains that produce non-planar deformations are ignored. Strictly, the BEBT fails for most practical considerations as shear

http://doi.org/10.30684/etj.2023.142505.1538
Received 10 August 2023; Received in revised form 05 October 2023; Accepted 06 October 2023; Available online 28 October 2023
20232412-0758/University of Technology-Iraq, Baghdad, Iraq

deformation is always present alongside flexural deformations. However, BEBT has been found to give excellent results for thin beams where shear deformations are insignificant compared to flexural deformation.

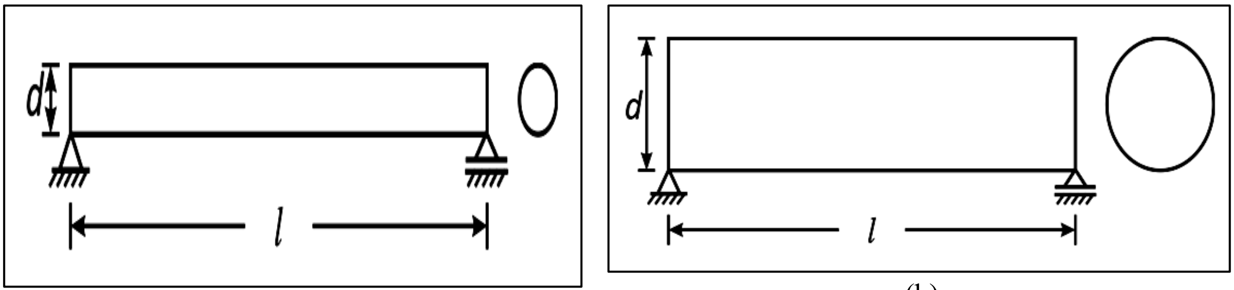
Nevertheless, structures such as thick beams, laminated composite, and sandwich beams have significant shear deformation that should be accounted for in their behavior. Timoshenko [1] pioneered the research on shear deformation, and his Timoshenko beam theory (TBT) has been extensively applied to flexural, buckling, and vibration problems. Elastic stability theory of various beam types and plates is presented by Timoshenko and Gere [2]. TBT, a first-order shear deformation theory (FSDT), considers the deformation to be the sum of flexural deformations wb and shear deformations ws Ike [3].

The TBT mited in producing constant shear strain across the thickness, thus violating the free shear strain requirement at the beam surfaces, Cowper [4]. Attempts to overcome the limitations of the TBT have led researchers to develop shear deformation beam theories that satisfy the transverse shear stress-free boundary conditions at the surfaces.



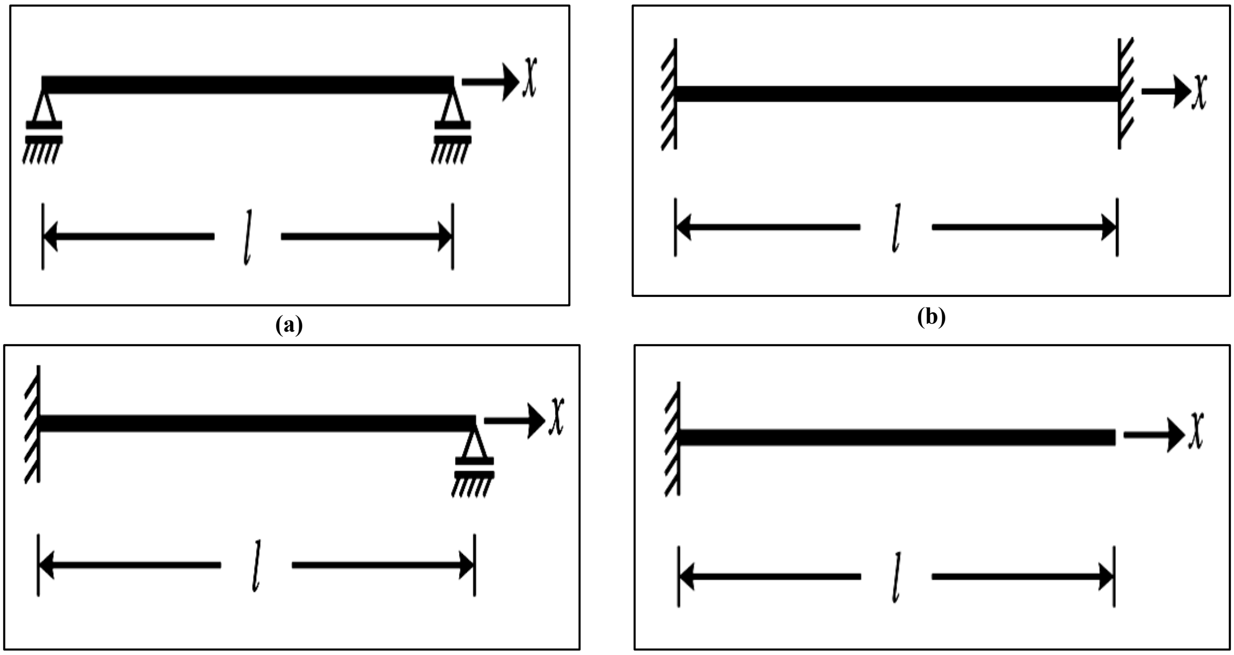
(a) **(b)**

Figure 1: (a)Thin beam (h/l < 0.05) with rectangular cross-section, (b) Thick beam with rectangular cross-section (h/l ≥ 0.05)



(a) (b)

Figure 2: (a)Thin beam with circular cross-section (d/l < 0.05),(b) Thick beam with circular cross-section (*d*/*l* ≥ 0.05)



(a) **(b)**

(c) **(d)**

Figure 3: (a) Simply supported beam (at x = 0, x = l), (b) Beam with clamped supports at x = 0, (c)x = l, Beam clamped at x = 0, simply supported at x = l, (d) Beam fixed at x = 0, free at x = l

Previous Studies

The foundations of the elastic stability theory of beams and plates are lucidly presented by Timoshenko and Gere [2]. Ghugal [5] developed a single variable parabolic shear deformation beam theory (PSDBT) for thick isotropic beams' static bending and dynamic bending vibrations. The formulation satisfied the transverse shear stress-free conditions at the plate surfaces and yielded stresses and displacements consistent with previous solutions from the literature. The formulation, however, did not consider beam buckling, which is the focus of this study.

Ghugal and Sharma [6] developed a hyperbolic shear deformation beam formulation for thick isotropic beams' static bending and dynamic bending. Their equations were variationally consistent and satisfied shear stress-free boundary conditions. They had stresses and displacement solutions that agreed with previously obtained solutions. Sayyad and Ghugal [7] also investigated the bending behavior of thick isotropic beams made of homogeneous materials using hyperbolic shear deformation beam theory (HSDBT). Their obtained solutions for stresses and displacements were in line with previous solutions in the literature. Ike [8] used the Fourier series method to determine analytical solutions for displacements and stress fields in thick beams whose governing equations were derived from first principles. The formulation yielded a hyperbolic shear deformable beam that satisfies the shear stress-free boundary conditions. The governing differential equations were obtained using the Euler-Lagrange equation on the total potential energy functional $\prod$. The study was validated by considering different transverse loadings, and the results agreed with previous literature. The work did not, however, consider the buckling of the thick beam. Ghugal and Shimpi [9] have presented a trigonometric shear deformation beam theory (TSDBT) for thick isotropic beams' flexural and natural vibration analysis. The TSDBT equations were formulated using the principle of virtual work as a set of differential equations in terms of displacements. The formulation satisfies the shear stress-free boundary conditions and is variationally consistent. Their solutions were in agreement with previous solutions. Krisha Murty [10] presented variationally consistent beam formulation using the principle of virtual work. Levinson [11] have presented a third-order shear deformation beam bending theory for rectangular thick beam. Ghugal [12], Sayyad [13], Ghugal and Shimpi [14], and Shimpi et al. [15] have studied refined beam theories for shear deformable rectangular beams made of isotropic, homogeneous materials. Their formulations were variationally consistent and satisfied shear stress-free conditions but failed to consider beam buckling. Sayyad and Ghugal [16] have presented a simple variable refined beam theory for thick beams' bending, natural vibrations, and stability analysis. Their equations were derived using the principle of virtual work and were variationally consistent satisfied the transverse shear stress-free conditions, and gave acceptable results.

Oguaghamba et al. [17] used the finite Fourier sine transformation method to solve the buckling problem of bisymmetric thin-walled beams with Dirichlet boundary conditions. In similar research, Oguaghamba et al. [18] used Ritz variational method to obtain buckling solutions to thin-walled beams with doubly symmetric cross-sections.

Onah et al. [19] and Mama et al. [20] have obtained closed-form analytical solutions to the buckling problem of moderately thick beams. The governing equation was solved using mathematical methods for solving differential equations, and exact buckling load solutions were obtained for beams with different boundary conditions. Heyliger and Reddy [21] used higher-order beam theories to obtain beam buckling solutions. Karama et al. [22] presented beam buckling solutions using exponential shear deformation beam buckling theory (ESDBBT).

Literature shows few studies have been done on buckling moderately thick and thick beams. This paper intends to fill this gap by formulating from first principles consideration the equations for the buckling of thick beams that consider a third-order polynomial transverse shear stress distribution across the thickness that also satisfies the transverse shear stress-free boundary conditions at the top and bottom surfaces of the beam. Literature also shows that the integral transform method has not been used to solve thick beam buckling problems. This paper applies the single finite sine transform method, an integral transformation technique, to develop buckling load solutions to the resulting governing equations of thick beam buckling.

2. Variational Formulation

2.1 Considered Thick Beam Buckling Problem

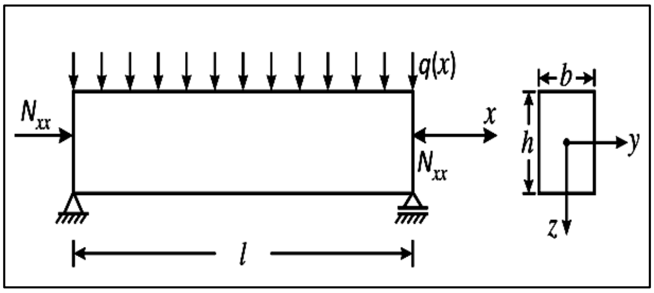


Figure 4: Simply supported thick rectangular beam

This paper considers the in-plane buckling of a simply supported thick beam of rectangular cross-section made of homogeneous, isotropic material, as shown in Figure 4. The beam cross-section has a depth of h, a width of b, and the beam has a span of l, as illustrated in Figure 4. The beam is defined: $0 \le x \le l$, $-b/2 \le y \le b/2$, $-h/2 \le z \le h/2$, where x, y, and z are the three-dimensional Cartesian coordinate axes.

2.2 Basic Assumptions

The assumptions are:

1) The x component of displacement is decomposed into two parts:
   a. A part is due to the *x* component of the analogous displacement of the classical Euler-Bernoulli beam bending theory.
   b. Another part is due to the transverse shear deformation, which varies across the thickness following the third-degree polynomial function.
2) The z component of the displacement uz depends on the *x* coordinate alone.
3) The materials stress–strain relationship follows the one-dimensional model and is linearly elastic, isotropic, and homogeneous.

2.3 Displacement Field

The displacement field expressions are Ghugal [5]:

$$u_x(x,z) = -z\frac{du_z(x)}{dx} - \left(\frac{1+\mu}{4}\right)h^2\beta(z)\frac{d^3u_z(x)}{dx^3} \tag{1}$$

$$u_y(x,z) = 0 \tag{2}$$

$$u_z(x,z) = u_z(x, z=0) = u_{z_0}(x) \tag{3}$$

where μ is Poisson's ratio, u_x, u_y, u_z, are the *x, y, z* components of the displacement, u_{z_0} is the transverse displacement field component at the neutral surface $z = 0$. $\beta(z)$ describes the distribution of transverse shear across the thickness. In Equation (1).

$$\beta(z) = z\left(1 - \frac{4}{3}\left(\frac{z}{h}\right)^2\right) = z - \frac{4}{3}\frac{z^3}{h^2} \tag{4}$$

2.4 Strain Field Expressions

From the small displacement linear elastic theory, the strain–displacement relations are used to find the strain field expressions:

$$\varepsilon_{xx} = \frac{du_x}{dx} = -z\frac{d^2u_{z_0}}{dx^2} - \left(\frac{1+\mu}{4}\right)h^2\beta(z)\frac{d^4u_{z_0}}{dx^4} \tag{5}$$

$$\varepsilon_{yy} = \frac{du_y}{dy} = 0 \tag{6}$$

$$\varepsilon_{zz} = \frac{du_z}{dz} = 0 \tag{7}$$

$$\gamma_{xz} = \frac{du_x}{dz} + \frac{du_z}{dx} = -\frac{du_{z_0}}{dx} + \frac{du_{z_0}}{dx} - \left(\frac{1+\mu}{4}\right)h^2\beta'(z)\frac{d^3u_{z_0}}{dx^3} = -\left(\frac{1+\mu}{4}\right)h^2\beta'(z)\frac{d^3u_{z_0}}{dx^3} \tag{8}$$

$$\gamma_{yz} = \frac{du_y}{dz} + \frac{du_z}{dy} = 0 \tag{9}$$

$$\gamma_{xy} = \frac{du_y}{dx} + \frac{du_x}{dy} = 0 \tag{10}$$

where ε_{xx}, ε_{yy}, and ε_{zz} are normal strains γ_{xy}, γ_{yz}, γ_{xz} are shear strains.

$$\beta'(z) = \frac{d\beta(z)}{dz} \tag{11}$$

2.5 Strains Fields

The normal stress σ_{xx}, σ_{yy}, σ_{zz} and shear stress τ_{xy}, τ_{xz}, τ_{yz} are obtained by using the one-dimensional stress-strain relations as follows:

$$\sigma_{xx} = E\varepsilon_{xx} = E\left(-z\frac{d^2u_{z_0}}{dx^2} - \left(\frac{1+\mu}{4}\right)h^2\beta(z)\frac{d^4u_{z_0}}{dx^4}\right) \tag{12}$$

$$\sigma_{yy} = E\varepsilon_{yy} = 0 \tag{13}$$

$$\sigma_{zz} = E\varepsilon_{zz} = 0 \tag{14}$$

$$\tau_{xz} = G\gamma_{xz} = -G\left(\frac{1+\mu}{4}\right)h^2\beta'(z)\frac{d^3u_{z_0}}{dx^3} \tag{15}$$

$$\tau_{yz} = G\gamma_{yz} = 0 \tag{16}$$

$$\tau_{xy} = G\gamma_{xy} = 0 \tag{17}$$

where E is Young's modulus, and G is the shear modulus.

2.6 Total Potential Energy Functional ∏

The total potential energy functional, ∏ is expressed by:

$$\Pi = U - V = \frac{1}{2}\iiint_V (\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz})dxdydz - \int_0^l u_{z_0}(x)q(x)\,dx - \frac{1}{2}\int_0^l N_{xx}\left(\frac{du_{z_0}(x)}{dx}\right)^2 dx \tag{18}$$

where U is the strain energy.

V is the potential energy of the beam due to external loads. The volume integration is carried out over the beam's three-dimensional (R^3) region. Substitution of the stress-strain laws gives.

$$\Pi = \frac{1}{2}\int_0^l\int_{-b/2}^{b/2}\int_{-h/2}^{h/2} (E\varepsilon_{xx}^2 + G\gamma_{xz}^2)\,dxdydz - \int_0^l u_{z_0}(x)q(x)\,dx - \frac{1}{2}\int_0^l N_{xx}\left(\frac{du_{z_0}(x)}{dx}\right)^2 dx \tag{19}$$

Hence, substituting expressions for ε_{xx} from Equation (5) and γ_{xz} from Equation (8), Equation (19) becomes:

$$\Pi = \frac{1}{2}\int_0^l\int_{-b/2}^{b/2}\int_{-h/2}^{h/2} E\left(-z\frac{d^2u_{z_0}(x)}{dx^2} - \left(\frac{1+\mu}{4}\right)h^2\beta(z)\frac{d^4u_{z_0}(x)}{dx^4}\right)dxdydz + $$
$$\frac{1}{2}\int_0^l\int_{-b/2}^{b/2}\int_{-h/2}^{h/2} G\left(-\left(\frac{1+\mu}{4}\right)h^2\beta'(z)\frac{d^3u_{z_0}}{dx^3}\right)dxdydz - \int_0^l u_{z_0}(x)q(x)\,dx - \frac{1}{2}\int_0^l N_{xx}\left(\frac{du_{z_0}}{dx}\right)^2 dx \tag{20}$$

Expanding gives:

$$\Pi = \frac{1}{2}E\int_0^l\int_{-b/2}^{b/2} dy\int_{-b/2}^{b/2}\left\{z^2\left(\frac{d^2u_{z_0}}{dx}\right)^2 + 2h^2\left(\frac{1+\mu}{4}\right)z\beta(z)\frac{d^2u_{z_0}}{dx^2}\frac{d^4u_{z_0}}{dx^4} + \left(\frac{1+\mu}{4}\right)^2 h^4(\beta(z))^2\left(\frac{d^4u_{z_0}}{dx^4}\right)^2\right\} +$$
$$\frac{1}{2}G\int_0^l\int_{-b/2}^{b/2} dy\int_{-h/2}^{h/2}\left\{\left(\frac{1+\mu}{4}\right)^2 h^4(\beta'(z))^2\left(\frac{d^3u_{z_0}}{dx^3}\right)^2\right\}dzdx - \int_0^l u_{z_0}(x)q(x)\,dx - \frac{1}{2}\int_0^l N_{xx}\left(\frac{du_{z_0}}{dx}\right)^2 dx \tag{21}$$

Further simplification gives:

$$\Pi = \frac{Eb}{2}\int_0^l \left\{ \int_{-h/2}^{h/2} z^2 dz \left(\frac{d^2u_{z_0}}{dx^2}\right)^2 dx + 2h^2\left(\frac{1+\mu}{4}\right)\left(\int_{-h/2}^{h/2} z\beta(z)\,dz\right)\frac{d^2u_{z_0}}{dx^2}\frac{d^4u_{z_0}}{dx^4}dx + \right.$$

$$\left.\left(\frac{1+\mu}{4}\right)^2 h^4\left(\int_{-h/2}^{h/2}(\beta(z))^2 dz\right)\left(\frac{d^4u_{z_0}}{dx^4}\right)^2 dx\right\} + \frac{Gb}{2}\int_0^l\left(\frac{1+\mu}{4}\right)^2 h^4\int_{-h/2}^{h/2}(\beta'(z))^2 dz\left(\frac{d^3u_{z_0}}{dx^3}\right)^2 dx -$$

$$\int_0^l u_{z_0}(x)q(x)\,dx - \frac{1}{2}\int_0^l N_{xx}\left(\frac{du_{z_0}}{dx}\right)^2 dx \tag{22}$$

Let $$I_1 = \int_{-h/2}^{h/2} z^2 dz \tag{23}$$

$$I_2 = \left(\frac{1+\mu}{4}\right)h^2\int_{-h/2}^{h/2} z\beta(z)\,dz \tag{24}$$

$$I_3 = \left(\frac{1+\mu}{4}\right)^2 h^4\int_{-h/2}^{h/2}(\beta(z))^2\,dz \tag{25}$$

$$I_4 = \left(\frac{1+\mu}{4}\right)^2 h^4\int_{-h/2}^{h/2}(\beta'(z))^2\,dz \tag{26}$$

Using Equations (23-26), Π is expressed as:

$$\Pi = \frac{Eb}{2}\int_0^l\left(I_1\left(\frac{d^2u_{z_0}}{dx^2}\right)^2 + 2I_2\frac{d^2u_{z_0}}{dx^2}\frac{d^4u_{z_0}}{dx^4} + I_3\left(\frac{d^4u_{z_0}}{dx^4}\right)^2\right)dx + \frac{Gb}{2}\int_0^l I_4\left(\frac{d^3u_{z_0}}{dx^3}\right)^2 dx - \int_0^l u_{z_0}q(x)\,dx - \frac{1}{2}\int_0^l N_{xx}\left(\frac{du_{z_0}}{dx}\right)^2 dx \tag{27}$$

were, $$G = \frac{E}{2(1+\mu)} \tag{28}$$

Substituting for G using Equation (28) gives Π as:

$$\Pi = \frac{Eb}{2}\int_0^l\left\{I_1\left(\frac{d^2u_{z_0}}{dx^2}\right) + 2I_2\frac{d^2u_{z_0}}{dx^2}\frac{d^4u_{z_0}}{dx^4} + I_3\left(\frac{d^4u_{z_0}}{dx^4}\right)^2 + \right.$$

$$\left.\frac{I_4}{2(1+\mu)}\left(\frac{d^3u_{z_0}}{dx^3}\right)^2 - \frac{2q(x)u_{z_0}}{Eb} + \frac{1}{Eb}N_{xx}\left(\frac{du_{z_0}}{dx}\right)^2\right\}dx \tag{29}$$

Equation (29) is of the general form:

$$\Pi = \frac{1}{2}Eb\int_0^l F(x, u_{z_0}, u'_{z_0}, u''_{z_0}, u'''_{z_0}, u^{iv}_{z_0})\,dx \tag{30}$$

where,

$$F = I_1(u''_{z_0})^2 + 2I_2 u''_{z_0} u^{iv}_{z_0} + I_3(u^{iv}_{z_0})^2 + \frac{I_4}{2(1+\mu)}(u'''_{z_0})^2 - \frac{2q(x)u_{z_0}}{Eb} - \frac{N_{xx}(u'_{z_0})^2}{Eb} \tag{31}$$

2.7 Euler-Lagrange Differential Equation (ELDE)

The ELDE is given by:

$$\frac{\partial F}{\partial u_{z_0}} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'_{z_0}}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial u''_{z_0}}\right) - \frac{d^3}{dx^3}\left(\frac{\partial F}{\partial u'''_{z_0}}\right) + \frac{d^4}{dx^4}\left(\frac{\partial F}{\partial u^{iv}_{z_0}}\right) = 0 \tag{32}$$

where:

$$\frac{\partial F}{\partial u_{z_0}} = -\frac{2q(x)}{Eb} \tag{33}$$

$$\frac{\partial F}{\partial u'_{z_0}} = -\frac{N_{xx}\cdot 2u'_{z_0}}{Eb} = -\frac{2N_{xx}u'_{z_0}}{Eb} \tag{34}$$

$$\frac{\partial F}{\partial u''_{z_0}} = 2u''_{z_0} I_1 + 2I_2 u^{iv}_{z_0} \tag{35}$$

$$\frac{\partial F}{\partial u'''_{z_0}} = \frac{I_4}{2(1+\mu)} 2u'''_{z_0} \tag{36}$$

$$\frac{\partial F}{\partial u^{iv}_{z_0}} = 2I_2 u''_{z_0} + 2I_3 u^{iv}_{z_0} \tag{37}$$

Substitution gives the Euler-Lagrange equation as:

$$-\frac{2q(x)}{Eb} - \frac{d}{dx}\left(\frac{-2N_{xx}u'_{z_0}}{Eb}\right) + \frac{d^2}{dx^2}(2u''_{z_0} I_1 + 2I_2 u^{iv}_{z_0}) - \frac{d^3}{dx^3}\left(\frac{I_4}{1+\mu}u'''_{z_0}\right) + \frac{d^4}{dx^4}(2I_2 u''_{z_0} + 2I_3 u^{iv}_{z_0}) = 0 \tag{38}$$

Hence,

$$-\frac{2q(x)}{Eb} + \frac{2N_{xx}}{Eb}u''_{z_0} + 2I_1 u^{iv}_{z_0} + 2I_2 u^{vi}_{z_0} - \frac{I_4}{1+\mu}u^{vi}_{z_0} + 2I_2 u^{vi}_{z_0} + 2I_3 u^{viii}_{z_0} = 0 \tag{39}$$

$$2I_3\frac{d^8 u_{z_0}}{dx^8} + \left(2I_2 + 2I_2 - \frac{I_4}{1+\mu}\right)\frac{d^6 u_{z_0}}{dx^6} + 2I_1\frac{d^4 u_{z_0}}{dx^4} + \frac{2N_{xx}}{Eb}\frac{d^2 u_{z_0}}{dx^2} = \frac{2q(x)}{Eb} \tag{40}$$

Simplifying by dividing by 2 gives:

$$I_3\frac{d^8 u_{z_0}}{dx^8} + \left(2I_2 - \frac{I_4}{2(1+\mu)}\right)\frac{d^6 u_{z_0}}{dx^6} + I_1\frac{d^4 u_{z_0}}{dx^4} + \frac{N_{xx}}{Eb}\frac{d^2 u_{z_0}}{dx^2} = \frac{q(x)}{Eb} \tag{41}$$

When no transverse distributed load is placed on the beam, $q(x)$ vanishes, and a homogeneous ordinary differential equation (ODE) results as follows:

$$I_3\frac{d^8 u_{z_0}}{dx^8} + \left(2I_2 - \frac{I_4}{2(1+\mu)}\right)\frac{d^6 u_{z_0}}{dx^6} + I_1\frac{d^4 u_{z_0}}{dx^4} + \frac{N_{xx}}{Eb}\frac{d^2 u_{z_0}}{dx^2} = 0 \tag{42}$$

3. Finite Sine Transform Method

Applying the finite sine transformation to Equation (42) gives:

$$\int_0^l \left(I_3 \frac{d^8 u_{z_0}}{dx^8} + \left(2I_2 - \frac{I_4}{2(1+\mu)} \right) \frac{d^6 u_{z_0}}{dx^6} + I_1 \frac{d^4 u_{z_0}}{dx^4} + \frac{N_{xx}}{Eb} \frac{d^2 u_{z_0}}{dx^2} \right) \sin \frac{n\pi x}{l} dx = 0 \tag{43}$$

By the linearity properties of the finite sine transformation, Equation (43) simplifies to:

$$I_3 \int_0^l \frac{d^8 u_{z_0}}{dx^8} \sin \frac{n\pi x}{l} dx + \left(2I_2 - \frac{I_4}{2(1+\mu)} \right) \int_0^l \frac{d^6 u_{z_0}}{dx^6} \sin \frac{n\pi x}{l} dx + I_1 \int_0^l \frac{d^4 u_{z_0}}{dx^4} \sin \frac{n\pi x}{l} dx + \frac{N_{xx}}{Eb} \int_0^l \frac{d^2 u_{z_0}}{dx^2} \sin \frac{n\pi x}{l} dx = 0 \tag{44}$$

By integration by parts and simplification using Dirichlet boundary conditions:

$$\int_0^l \frac{d^8 u_{z_0}}{dx^8} \sin \frac{n\pi x}{l} dx = \left(\frac{n\pi}{l} \right)^8 \int_0^l u_{z_0} \sin \frac{n\pi x}{l} dx = \left(\frac{n\pi}{l} \right)^8 \bar{U}_{z_0}(n) \tag{45}$$

$$\int_0^l \frac{d^6 u_{z_0}}{dx^6} \sin \frac{n\pi x}{l} dx = -\left(\frac{n\pi}{l} \right)^6 \int_0^l u_{z_0} \sin \frac{n\pi x}{l} dx = -\left(\frac{n\pi}{l} \right)^6 \bar{U}_{z_0}(n) \tag{46}$$

$$\int_0^l \frac{d^4 u_{z_0}}{dx^4} \sin \frac{n\pi x}{l} dx = \left(\frac{n\pi}{l} \right)^4 \int_0^l u_{z_0} \sin \frac{n\pi x}{l} dx = \left(\frac{n\pi}{l} \right)^4 \bar{U}_{z_0}(n) \tag{47}$$

$$\int_0^l \frac{d^2 u_{z_0}}{dx^2} \sin \frac{n\pi x}{l} dx = -\left(\frac{n\pi}{l} \right)^2 \int_0^l u_{z_0} \sin \frac{n\pi x}{l} dx = -\left(\frac{n\pi}{l} \right)^2 \bar{U}_{z_0}(n) \tag{48}$$

Hence,

$$I_3 \left(\frac{n\pi}{l} \right)^8 \bar{U}_{z_0}(n) - \left(\frac{n\pi}{l} \right)^6 \left(2I_2 - \frac{I_4}{2(1+\mu)} \right) \bar{U}_{z_0}(n) + \left(\frac{n\pi}{l} \right)^4 I_1 \bar{U}_{z_0}(n) - \left(\frac{n\pi}{l} \right)^2 \frac{N_{xx}}{Eb} \bar{U}_{z_0}(n) = 0 \tag{49}$$

Simplifying further,

$$\left(I_3 \left(\frac{n\pi}{l} \right)^8 - \left(\frac{n\pi}{l} \right)^6 \left(2I_2 - \frac{I_4}{2(1+\mu)} \right) + \left(\frac{n\pi}{l} \right)^4 I_1 - \left(\frac{n\pi}{l} \right)^2 \frac{N_{xx}}{Eb} \right) \bar{U}_{z_0}(n) = 0 \tag{50}$$

4. Results and Discussion

4.1 Results

For non-trivial solutions

$$\bar{U}_{z_0}(n) \neq 0$$

Hence, the characteristic buckling equation becomes:

$$I_3 \left(\frac{n\pi}{l} \right)^8 - \left(\frac{n\pi}{l} \right)^6 \left(2I_2 - \frac{I_4}{2(1+\mu)} \right) + \left(\frac{n\pi}{l} \right)^4 I_1 - \frac{N_{xx}}{Eb} \left(\frac{n\pi}{l} \right)^2 = 0 \tag{51}$$

Solving for N_{xx},

$$N_{xx}=\frac{I_3\left(\frac{n\pi}{l}\right)^8-\left(\frac{n\pi}{l}\right)^6\left(2I_2-\frac{I_4}{2(1+\mu)}\right)+\left(\frac{n\pi}{l}\right)^4 I_1}{\frac{1}{Eb}\left(\frac{n\pi}{l}\right)^2} \quad (52)$$

Simplifying Equation (52) gives:

$$N_{xx}=Eb\left(\frac{l}{n\pi}\right)^2\left(I_3\left(\frac{n\pi}{l}\right)^8-\left(\frac{n\pi}{l}\right)^6\left(2I_2-\frac{I_4}{2(1+\mu)}\right)+\left(\frac{n\pi}{l}\right)^4 I_1\right) \quad (53)$$

Further simplification of Equation (53) gives:

$$N_{xx}=Eb\left(I_3\left(\frac{n\pi}{l}\right)^6-\left(\frac{n\pi}{l}\right)^4\left(2I_2-\frac{I_4}{2(1+\mu)}\right)+\left(\frac{n\pi}{l}\right)^2 I_1\right) \quad (54)$$

Substituting the expressions for the integrals gives:

$$N_{xx}=Eb\left(\frac{17h^7}{315}\left(\frac{1+\mu}{4}\right)^2\left(\frac{n\pi}{l}\right)^6-\left(\frac{n\pi}{l}\right)^4\left(\frac{2(1+\mu)h^5}{60}-\frac{h^5(1+\mu)^2}{2\times 30(1+\mu)}\right)+\left(\frac{n\pi}{l}\right)^2\frac{h^3}{12}\right) \quad (55)$$

Further simplification of Equation (55) gives:

$$N_{xx}=Ebh^3\left(\frac{n\pi}{l}\right)^2\left(\frac{17}{315}\frac{(1+\mu)^2}{16}(n\pi)^4\left(\frac{h}{l}\right)^4-(n\pi)^2\left(\frac{(1+\mu)h^2}{30}-\frac{h^2(1+\mu)}{60}\right)\left(\frac{h}{l}\right)^2+\frac{1}{12}\right) \quad (56)$$

Further simplification of Equation (56) gives:

$$N_{xx}=\frac{Ebh^3}{l^2}(n\pi)^2\left(\frac{17(1+\mu)^2}{5040}(n\pi)^4\left(\frac{h}{l}\right)^4-(n\pi)^2\left(\frac{1+\mu}{60}\right)\left(\frac{h}{l}\right)^2+\frac{1}{12}\right) \quad (57)$$

N_{xx} is critical when $n=1$.
Hence,

$$N_{xx}(n=1)=N_{xx\,cr}=\frac{Ebh^3}{l^2}\pi^2\left(\frac{17(1+\mu)^2}{5040}\pi^4\left(\frac{h}{l}\right)^4-\pi^2\left(\frac{1+\mu}{60}\right)\left(\frac{h}{l}\right)^2+\frac{1}{12}\right) \quad (58)$$

Nxx cr is calculated using Equation (58) for $\frac{h}{l}=\frac{1}{4}$, $\frac{h}{l}=\frac{1}{10}$, $\frac{h}{l}=\frac{1}{100}$, and presented in Table 1 in the dimensionless form:

$$N_{xx\,cr}=\frac{Ebh^3}{l^2}K_{cr} \quad (59)$$

where

$$K_{cr}=\pi^2\left(\frac{17(1+\mu)^2\pi^4}{5040}\left(\frac{h}{l}\right)^4-\frac{(1+\mu)}{60}\pi^2\left(\frac{h}{l}\right)^2+\frac{1}{12}\right) \quad (60)$$

$$\text{Then } \frac{N_{xx\,cr}l^2}{Ebh^3}=K_{cr} \quad (61)$$

K_{cr} is a simply supported thick beam's dimensionless critical buckling load coefficient.

Values of K_{cr} for h/l ranging from 0.01 to $h/l=0.25$ are presented in Table 1 and compared with values of K_{cr} obtained from previous research studies using exponential shear deformation beam theory (ESDT), higher-order shear deformation theory (HSDT) and Timoshenko [1] beam theory (FSDT).

Table 1: Comparative study of Kcr for simply supported thick beam

K_{cr}			
$h/l = 0.25$ **(Thick beam) (l/h = 4)**	$h/l = 0.10$ **(Moderately thick beam) (l/h = 10)**	$h/l = 0.010$ **(thin beam) (l/h = 100)**	**Ref.**
0.71196631	0.80190976	0.822256035	Present
0.8225	0.8225	0.8225	CBT [2]
–	–	–	Exact [4]
0.7088	0.8019	0.8222	FSDT [1]
0.7090	0.8019	0.8220	HSDBT [21]
0.7116	0.8020	0.8223	ESDBT [22]
0.7112	0.8018	0.8223	PSDT [16]

For $\mu = 0.30$

$$K_{cr} = \pi^2\left(\frac{17(1.3)^2\pi^4}{5040}\left(\frac{h}{l}\right)^4 - \frac{1.3\pi^2}{60}\left(\frac{h}{l}\right)^2 + \frac{1}{12}\right) \tag{62}$$

Using Equation (62), Table 2 is presented for the variation of K_{cr} as l/h values.

Table 2: Table of values for Kcr (when $\mu = 0.30$) for various values of l/h ranging from 4 to 100 for cubic polynomial shear deformation beams

l/h	Kcr (to 4 decimal places)
4	0.7120
6	0.7681
8	0.7908
10	0.8019
15	0.8132
20	0.8172
40	0.8212
50	0.8216
60	0.8219
80	0.8221
100	0.8223

Figure 5 shows the graph of Kcr vs. l/h for the cubic polynomial shear deformation beam solved in this paper. Figure 5 is drawn using the table of values shown in Tables 1 and 2 and also shows the graph of Kcr vs l/h using Table 1 for results obtained by Timoshenko [1], Timoshenko and Gere [2], Cowper [4], Heyliger and Reddy [21], Karama et al. [22] and Sayyad and Ghugal [16]. Figure 5 illustrates the very close agreement between the present finite transform results and those obtained in the literature.

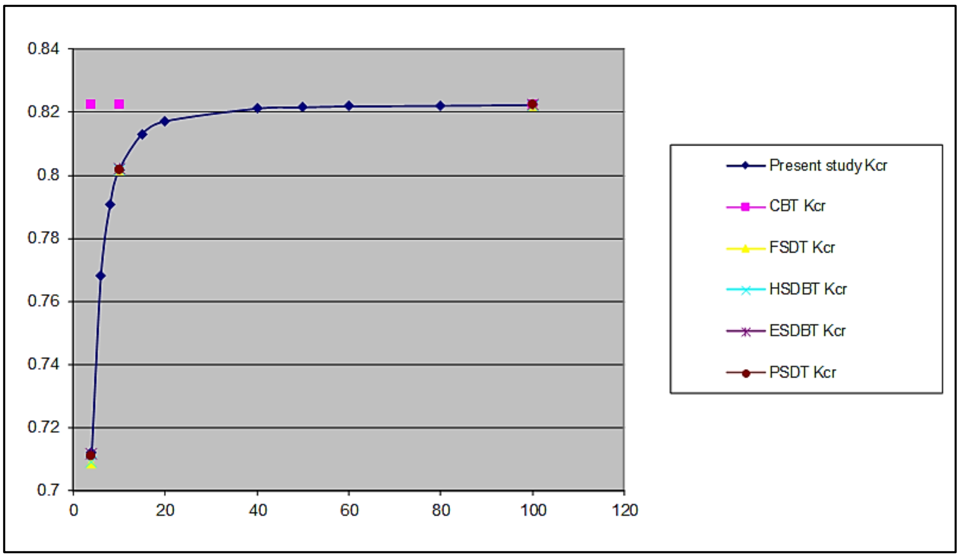


Figure 5: Graph of Kcr vs l/h

4.2 Discussion

This paper has presented the finite sine transform method for solving the buckling of thick, homogeneous, isotropic rectangular beams for the first time. The governing domain equations were formulated from the first principles using variational calculus. Transverse shear deformation was considered in the thick beam formulation by assuming that the x component of the displacement field comprises a component due to bending deformation and another complement due to shear deformation. The distribution of transverse shear stress across the thickness is constructed to ensure that shear stress-free boundary conditions at the top and bottom surfaces are satisfied, thus preventing the need for shear correction factors of the first-order shear deformation theories (FSDTs).

The total potential energy functional $\prod$ is found as a functional dependent upon the transverse deflection of the neutral surface u_{z_0} of the beam, and derivatives of u_{z_0}. Euler-Lagrange differential equations are applied to the integrand in $\prod$ to find the differential equations of stability of the thick beam problem as an eight order linear ordinary differential equation which in non-homogeneous when $q(x)$ is present, and homogeneous when there is no transverse load $q(x)$.

For buckling problems in the absence of transverse load, the finite sine transformation of the eight-order ODE results in a transformation of the problem to the algebraic eigenvalue problem represented in Equation (51). The solution of Equation (51) for the buckling loads at any arbitrary buckling mode, n, is found in Equation (57). The critical buckling load $N_{xx\,cr}$ is the least buckling load that occurs at the first mode $n = 1$, and is found in Equation (58), which is presented in terms of dimensionless critical buckling load coefficients K_{cr} as Equations (59) and (60).

Values of K_{cr} for $h/l = 0.25$, $h/l = 0.10$ and $h/l = 0.010$ are presented in Table 1, together with previously published values of K_{cr} for FSDT, HSDT, ESDT, PSDT, and CBT. Table 1 shows that for $h/l = 0.01$, which is for thin beams, K_{cr} obtained in the present study agrees exactly with the results obtained by Sayyad and Ghugal [16], who used a polynomial shear deformation theory PSDT to solve the candidate problem. The results are also identical to results obtained using CBT (or BEBT), indicating that the contribution of shear deformation to the buckling behavior is insignificant when $h/l = 0.01$. The results generally agree with results by Cowper [4], Timoshenko [1], Heyliger, and Reddy [21] and the ESDT.

Table 1 further shows that at $h/l = 0.10$, $(l/h = 10)$ which is a moderately thick beam $K_{cr} = 0.80191$ obtained by this present method is 2.50% smaller than the value from the BEBT. This shows that at $h/l = 0.10$, the shear deformation has some effect in reducing the load-carrying capacity of the moderately thick beam. The implication is that the BEBT (CBT) overestimates the critical buckling load of moderately thick beams and cannot be safely used in their analysis and design. The value obtained for K_{cr} when $h/l = 0.10$, agrees with the K_{cr} values obtained using ESDBT, HSDBT, ESDBT, and PSDBT.

Similarly, for $h/l = 0.25$, $(l/h = 4)$ which correspond to thick beams, the $K_{cr} = 0.71196631 \approx 0.7120$ value obtained by this formulation is 13.44% lower than the K_{cr} for BEBT. This implies that the shear deformation has greatly contributed to considerably reducing the critical load-carrying capacity of thick beams. The values of K_{cr} for $h/l = 0.25$ in this study agree with previous values for K_{cr} found by Cowper [4], Timoshenko [1], Heyliger and Reddy [21], and Sayyad and Ghugal [16].

5. Conclusion

This study has presented a rigorous first-principles formulation and derivation of the governing equation of stability of a thick beam. The formulation is variationally constant and considers transverse shear stress variation through the thickness in a cubic function such that transverse shear stress-free boundary conditions are satisfied at the top and bottom surfaces. The equation of buckling is an eight-order ODE, which is non-homogeneous when $q(x) \neq 0$, and homogeneous when $q(x) = 0$.

- The finite sine transformation of the governing ODE ($q(x) = 0$) simplifies the problem to an algebraic eigenvalue problem.
- The solution of the algebraic problem gives the buckling loads at any buckling mode, n.
- The critical buckling load $N_{xx\,cr}$ occurs at the first buckling mode $n = 1$, and is found as an analytical closed-form expression dependent upon h/l.
- $N_{xx\,cr}$ is generally expressed in terms of K_{cr}, the critical buckling load coefficient, which depends upon h/l.
- For $h/l = 0.01$, (thin beam), K_{cr} found was identical to K_{cr} for BEBT, indicating that shear deformation effects have a negligible contribution to the critical buckling load, N_{xxcr}.
- For $h/l = 0.10$, (moderately thick beams), K_{cr} was found to be 2.50% lower than the K_{cr} for BEBT, indicating that the transverse shear deformation effect significantly lowers the critical buckling load for moderately thick beams.
- For $h/l = 0.25$, (thick beams), K_{cr} was found to be 13.44% lower than the K_{cr} for BEBT, indicating the huge risks involved in using BEBT to analyze and design thick beams and the attendant overestimate of critical buckling load capacities associated with BEBT.

Notation/Nomenclature

x axial/longitudinal coordinate of the beam, y coordinate of the beam along the width
z transverse coordinate of beam
d diameter of beam cross-section
h thickness of the beam
b width of the beam
l length (span) of beam

n buckling node number
Π total potential energy functional
$N_{xx\,cr}$ critical buckling load
$q(x)$ transverse load distribution
μ Poisson's ratio
u_x displacement component in x direction
u_y y component of displacement
u_z z component of displacement
$\beta(z)$ distribution of transverse shear stress across the thickness
u_{z_0} transverse displacement of the neutral surface $(z = 0)$ of the beam
ε_{xx} normal strain in x direction
ε_{yy} normal strain in y direction
ε_{zz} normal strain in z direction
γ_{xy} shear strain in xy plane
γ_{xz} shear strain in xz plane
γ_{yz} shear strain in yz plane
E Young's modulus of elasticity
G shear modulus
U strain energy of flexural deformation
V potential energy of beam due to external loads
R^3 three-dimensional region of integration
I_1, I_2, I_3, I_4 integrals expressed in terms of z, $\beta(z)$ and $\beta'(z)$
F integrand in the functional Π
$\frac{\partial}{\partial u_{z_0}}$ partial u_{z_0} derivative concerning
$\frac{d^r}{dx^r}$ rth ordinary derivative concerning x
% percent
CPSDBBT cubic polynomial shear deformation beam buckling theory
ELDE Euler-Lagrange differential equation
CTBT classical thin beam theory
BEBT Bernoulli-Euler beam theory
TSDBT trigonometric shear deformation beam theory
ESDBT exponential shear deformation beam theory
HSDBT higher order shear deformation beam theory
PSDT polynomial shear deformation theory
PSDBT polynomial shear deformation beam theory
ODE ordinary differential equation
$\int(\ )dx$ integrationconcerningo x; integral
$\bar{U}_{z_0}(n)$ finite Fourier sine transform of $u_{z_0}(x)$
K_{cr} critical buckling load coefficient
ESDBBT exponential shear deformation beam buckling theory.

Funding

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

Data availability statement

The data that support the findings of this study are available on request from the corresponding author.

Conflicts of interest

The authors declare that there is no conflict of interest.

References

[1] S .P. Timoshenko, On the correction for shear of the differential equation for transverse vibrations of prismatic bars, Philos. Mag., 41 (1921) 742-746. https://doi.org/10.1080/14786442108636264

[2] S. P. Timoshenko, G.M. Gere Theory of elastic stability 2nd Edition, McGraw Hill Book Company, New York,1961.

[3] C. C. Ike Timoshenko beam theory for the flexural analysis of moderately thick beams-variational formulation and closed-form solutions, Ital. J. Eng. Sci., 63 (2019) 34-45. http://dx.doi.org/10.18280/ti-ijes.630105

[4] G. R. Cowper On the accuracy of Timoshenko beam theory, ASCE J. Eng. Mech., 94 (1968) 1447-1453.

[5] Y. M. Ghugal, A single variable parabolic shear deformation theory for flexure and flexural vibration of thick isotropic beams, Proc. Third Intl Conf. Struct. Eng., Mech. Comput., Cape Town, South Africa, 2007, 77-78.

[6] Y. M. Ghugal R. Sharmam, A hyperbolic shear deformation theory for flexure and vibration of thick isotropic beams, Int. J. Comput Methods, 6 (2009) 585-604. https://doi.org/10.1142/S0219876209002017

[7] A.S. Sayyad, Y.M. Ghugal, Flexure of thick beams using hyperbolic shear deformation theory, Int. J. Mech. 5 (2011) 111-122.

[8] C.C. Ike, Fourier series method for finding displacements and stress fields in hyperbolic shear deformable thick beams subjected to distributed transverse loads, J. Comput. Appl. Mech., 53 (2022) 126-141. https://doi.org/10.22059/jcamech.2022.332719.658

[9] Y. M. Ghugal, R.P. Shimpi, A Trigonometric shear deformation theory for flexure and free vibration of thick isotropic beams. Proc. Struct. Eng. Convention (SEC2000), IT Bombay, Mumbai, India, 2000, 255-263,

[10] A.V. Krishna Murty, Towards a consistent beam theory, AIAA J., 22 (1984) 811-816.

[11] M. Levinson, A new rectangular beam theory, J. Sound Vib., 74 (1984) 81-87. https://doi.org/10.1016/0022-460X(81)90493-4

[12] Y. M. Ghugal, A new refined bending theory for thick beam including transverse shear and transverse normal strain effects. Departmental Report, Applied Mechanics Department, Government College of Engineering, Avrangabad, India, 2006, 1 – 96.

[13] A. S. Sayyad, Comparison of various refined beam theories for the bending and free vibration analysis of thick beams, Appl. Comput. Mech., 5 (2011) 217-230.

[14] Y. M. Ghugal, R.P. Shimpi, A review of refined shear deformation theories for isotropic and anisotropic beams, J. Reinf. Plast. Compos., 20 (2001) 255-272. https://doi.org/10.1177/073168401772678283

[15] R. P. Shimpi, P.J. Gurprasad, K.S. Pakhere, Simple two variable refined theory for shear deformable isotropic rectangular beams, J. Appl. Comput. Mech., 6 (2019) 394-415. https://doi.org/10.22055/jacm.2019.29555.1615

[16] A. S. Sayyad, Y.M. Ghugal, Single variable refined beam theories for the bending, buckling and free vibration of homogeneous beams, Appl. Comput. Mech. 10 (2016) 123-138

[17] O. A. Oguaghamba, C.C. Ike, E.U. Ikwueze, I.O. Ofondu, Finite Fourier sine integral transform method for the elastic buckling analysis of doubly symmetric thin-walled beams with Dirichlet boundary conditions, ARPN J. Eng. Appl. Sci., 14 (2019) 3968-3974.

[18] O. A. Oguaghamba, C.C. Ike, E.U. Ikwueze, I.O. Ofondu, Ritz variational method for solving the elastic buckling problems of thin-walled beams with bisymmetric cross-sections, Math. Model. Eng. Probl., 10 (2023) 129-139. https://doi.org/10.18280/mmep.100114

[19] H. N. Onah, C.U. Nwoji, M.E. Onyia, B.O. Mama, C.C. Ike. Exact solutions for the elastic buckling problem of moderately thick beam, J. Compos. Adv. Mater., 30 (2020) 83-93. https://doi.org/10.18280/rcma.300205

[20] B.O. Mama, O.A. Oguaghamba, C.C. Ike, Finite Fourier sine transform method for finding natural frequencies of flexural vibration of moderately thick beams, Proc. 2020 Sustainable Eng. Ind. Tech. Conf., 2020, A5-1 – A5-5.

[21] P. R. Heyliger, J.N. Reddy, A higher order beam finite element for bending and variation problems, J. Sound Vib. 126 (1988) 309-326. https://doi.org/10.1016/0022-460X(88)90244-1

[22] M. Karama, K.S. Afaq, S. Mistou, Mechanical behaviour of laminated composite beam by new multi-layered laminated composite structures model with transverse shear stress continuity, Int. J. Solids Struct., 40 (2003) 1525-1546. https://doi.org/10.1016/S0020-7683(02)00647-9

Appendix 1

$$b(z) = z - \frac{4}{3}\frac{z^3}{h^2}$$

$$\beta'(z) = \left(1 - \frac{4}{3}\frac{3z^2}{h^2}\right) = 1 - \frac{4z^2}{h^2}$$

$$I_1 = \int_{-h/2}^{h/2} z^2 \, dz = \left.\frac{z^3}{3}\right]_{-h/2}^{h/2} = \frac{1}{3}\left(\left(\frac{h}{2}\right)^3 - \left(\frac{-h}{3}\right)^3\right)$$

$$I_1 = \frac{1}{3}\left(\frac{h^3}{8} + \frac{h^3}{8}\right) = \frac{h^3}{12}$$

$$I_2 = \left(\frac{1+\mu}{4}\right) h^2 \int_{-h/2}^{h/2} z\left(z - \frac{4}{3}\frac{z^3}{h^2}\right) dz$$

$$I_2 = \left(\frac{1+\mu}{4}\right) h^2 \int_{-h/2}^{h/2} \left(z^2 - \frac{4}{3}\frac{z^4}{h^2}\right) dz$$

$$I_2 = \left(\frac{1+\mu}{4}\right) h^2 \left[\frac{z^3}{3} - \frac{4}{3}\frac{z^5}{5h^2}\right]_{-h/2}^{h/2}$$

$$I_2 = \left(\frac{1+\mu}{4}\right) h^2 \left[\frac{z^3}{3} - \frac{4}{15h^2} z^5\right]_{-h/2}^{h/2}$$

$$I_2 = \left(\frac{1+\mu}{4}\right) h^2 \left(\frac{1}{3}\left(\frac{h}{2}\right)^3 - \frac{4}{15h^2}\left(\frac{h}{2}\right)^5\right) \times 2$$

$$I_2 = \left(\frac{1+\mu}{4}\right) h^2 h^3 \left(\frac{1}{24} - \frac{1}{120}\right) \times 2$$

$$I_2 = \left(\frac{1+\mu}{4}\right) \frac{h^5}{15} = \frac{(1+\mu)h^5}{60}$$

$$I_3 = \left(\frac{1+\mu}{4}\right)^2 h^4 \int_{-h/2}^{h/2} \left(z - \frac{4}{3}\frac{z^3}{h^2}\right)^2 dz$$

$$I_3 = \left(\frac{1+\mu}{4}\right)^2 h^4 \int_{-h/2}^{h/2} \left(z^2 - \frac{2\times 4}{3}\frac{z^4}{h^2} + \frac{16}{9}\frac{z^6}{h^4}\right) dz$$

$$I_3 = \left(\frac{1+\mu}{4}\right)^2 h^4 \left[\frac{z^3}{3} - \frac{8}{3h^2}\frac{z^5}{5} + \frac{16}{9h^4}\frac{z^7}{7}\right]_{-h/2}^{h/2}$$

$$I_3 = \left(\frac{1+\mu}{4}\right)^2 h^4 \left[\frac{z^3}{3} - \frac{8}{15h^2} z^5 + \frac{16}{63h^4} z^7\right]_{-h/2}^{h/2}$$

$$I_3 = \left(\frac{1+\mu}{4}\right)^2 h^4 \left(\frac{z^3}{3} - \frac{8}{15h^2} z^5 + \frac{16}{63h^4} z^7\right)$$

$$I_3 = \left(\frac{1+\mu}{4}\right)^2 h^4 \left(\frac{1}{3}\left(\frac{h}{2}\right)^3 - \frac{8}{15h^2}\left(\frac{h}{2}\right)^5 + \frac{16}{63h^4}\left(\frac{h}{2}\right)^7\right) \times 2$$

$$I_3 = \left(\frac{1+\mu}{4}\right)^2 h^4 \left(\frac{h^3}{24} - \frac{h^3}{60} + \frac{h^3}{504}\right) \times 2$$

$$I_3 = \left(\frac{1+\mu}{4}\right)^2 h^4 \left(\frac{136h^3}{2520}\right) = \left(\frac{1+\mu}{4}\right)^2 h^4 \left(\frac{17h^3}{315}\right)$$

$$I_3 = \left(\frac{1+\mu}{4}\right)^2 h^7 \left(\frac{17}{315}\right) = \frac{17h^7}{315}\left(\frac{1+\mu}{4}\right)^2$$

$$I_4 = \left(\frac{1+\mu}{4}\right)^2 h^4 \int_{-h/2}^{h/2} \left(1 - \frac{4}{h^2} z^2\right)^2 dz$$

$$I_4 = \left(\frac{1+\mu}{4}\right)^2 h^4 \int_{-h/2}^{h/2} \left(1 - \frac{8}{h^2} z^2 + \frac{16}{h^4} z^4\right) dz$$

$$I_4 = \left(\frac{1+\mu}{4}\right)^2 h^4 \left[z - \frac{8}{h^2}\frac{z^3}{3} + \frac{16}{h^4}\frac{z^5}{5}\right]_{-h/2}^{h/2}$$

$$I_4 = \left(\frac{1+\mu}{4}\right)^2 h^4 \left[\frac{h}{2} - \frac{8}{3h^2}\left(\frac{h}{2}\right)^3 + \frac{16}{5h^4}\left(\frac{h}{2}\right)^5\right] \times 2$$

$$I_4 = \left(\frac{1+\mu}{4}\right)^2 h^4 h\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{10}\right) \times 2 = \left(\frac{1+\mu}{4}\right)^2 h^5 \cdot \frac{16}{30} \; I_4 = \frac{8}{15} h^5 \left(\frac{1+\mu}{4}\right)^2 = \frac{h^5(1+\mu)^2}{30}$$