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Solving Westergaard Half-Space Problems Using Potential Theory

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HIGHLIGHTS

- The governing differential equation is derived for an elastic half-space problem with horizontal inextensibility.
- The potential theory is applied to solve the Westergaard problem for a point load on the boundary.
- The approach adopts first principles to derive the governing equations, highlighting limitations and scope.
- Results are validated by comparison with literature sources.

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ABSTRACT

The Westergaard half-space problem has been solved using the potential theory in this work. It is a classical theme in elasticity theory that seeks to find the displacements and stresses in the half-space caused by known boundary loads. It has important applications in analyzing stresses and displacement fields in soil due to applied points and distributed loads on the boundary caused by structures placed on the soil. It is governed by stress–strain, strain-displacement, and equilibrium equations. Horizontal inextensibility is assumed in developing the problem, simplifying the displacement formulation to a three-dimensional (3D) Laplace equation. The potential theory is applied to find the vertical displacement. Stress–displacement equations obtained from the simultaneous use of the kinematic and stress-strain equations are used to obtain the stress fields. The specific problem of point load at the origin was considered and solved. The equilibrium of internal vertical stresses and the external vertical load is used to find the integration constant. Hence, vertical displacements were found. The stress fields were found from the stress–displacement equations. The expressions for the vertical displacements and stresses were found to be exact within the framework of the theory used, as they satisfied all the governing equations of the problem. However, the solutions become unbounded at the origin due to the singularity of the vertical displacement and stresses. The obtained solutions are identical to previously obtained solutions.

1. Introduction

Elastic half-space problems are problems in the mathematical theory of elasticity concerned with the determination of stresses and displacements in the three-dimensional (3D) semi-infinite medium due to loads assumed concentrated or distributed, which may be applied either on the boundary surface or in the semi-infinite medium [1–3]. Thus, elastic half-space problems involve 3D media defined over the region where the plane is the boundary plane [4 – 6]. When the vertical point load is applied on the surface, the problem is described as a Boussinesq problem [7– 9]. When a horizontal point load is applied to the surface, the problem is called a Cerrutti problem [8–10].

The classical problem in the mathematical theory of elasticity of finding fields of stresses and displacements in an isotropic, homogeneous, semi-infinite linear elastic medium subjected to vertical concentrated loads applied on the boundary was first considered, analyzed, and determined by Boussinesq [11–13].

Boussinesq developed analytical solutions to the problem by using Green's function of the three-dimensional Laplace equation to determine the stress equilibrium fields in the semi-infinite elastic medium [14 – 16]. Boussinesq further obtained solutions for the distributions of stresses and displacements in the half-space using the classical theory of linear small displacement elasticity and potential theory [17–20].

Chan [20] presented a seminal work on analytical methods and illustrated their applications to the solutions of geomechanics problems. Ferretti [21] studied homogeneous linear elastic and isotropic half-spaces under loads applied perpendicularly to the boundary surface and particularly studied distributed loadings on such boundary surfaces. Sadd [22] presented the fundamental

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principles of the elasticity formulation of elastic half-space problems. Bowles [23] presented elastic half-space problems and vividly studied the Boussinesq and Westergaard stress distribution theories of elastic half-space. Further studies of the elastic half-space problems of Kelvin, Boussinesq, Flammant, Cerrutti, Melan, and Mindlin are found in Podio-Guildguli and Favata [24] and also in Sitheram and Gounda-Reju [25]. Teodorescu [26] and Davis and Salvadurai [27] published a treatise on the theory of elasticity and its application to geomechanics and geotechnical problems. Abeyartne [28] and Zhon and Gao [29] studied spatial problems of elastic half-space.

Khapilova and Zaletov [30] derived mathematical expressions for the fields of stresses and displacements in an axially symmetrical elastic half-space problem with the boundary fixed elastically outside the circular loaded area. They then used their obtained expressions for distributed loads over circular areas on the half-space to formulate analytical expressions for point load on an isotropic elastic half-space with a fixed surface. They found that in the limiting case where the coefficient of proportionality is zero, their obtained analytical expressions reduced to the analytical expressions previously obtained as solutions for stresses and displacements for the Boussinesq problem of vertical point load on elastic half-space.

Other significant research works on elastic half-space theory are found in Morshedifard and Eskandari-Ghadi [31], Tekinsay et al. [32], Anyaegbunem et al [33], Ojedokun and Olutoge [34], Westergaard [35], Ike [36 – 44], Ike et al. [45, 46] and Onah et al [47].

Morshedifard and Sakandari-Ghadi [31] studied transversely isotropic elastic half-space under loading from flexible structure using coupled boundary element – finite element silone and assuming three-dimensional dynamic interactions. Tekinsey et al. [32] presented approximate solutions for stress fields for non-isotropic elastic half-space made of clay soil.

Ojedokun and Olutoge [34] applied Bousssinesq and Westergaard's stress distribution theories to investigate the failure of the foundation of a collapsed telecommunication mast. Anyaegbunam et al. [33] published an article in which they argued the non-existence of the Westergaard stresses solution for point load acting on the boundary of a horizontally rigid half-space, also termed the Westergaard half-space. In their paper, they showed that Westergaard's solutions for stresses do not satisfy one of the stress boundary conditions: the boundary should be free of shear stresses. They thus argued that the Westergaard theory does not give the exact solution for a point load in the surface of a Westergaard half space.

Westergaard [35] assumed zero value for the Poisson's ratio of the soil continuum to completely present lateral strain and thus allow only vertical deformation of the soil half-spaces. Westergaard [35] consequently developed, for the first time, vertical stress equations applicable to a soil continuum with simplified assumptions due to a point load acting on the boundary of the continuum.

Ike [36] used Elzaki transformation method to find closed-form solutions to elastic half-plane problems in polar coordinates using Airy stress function of expressing the 2D elasticity problem. Ike [37] also used Fourier integral transform method to obtain closed-form solutions to 2D elastic half-plane problems using Love stress function approach. Ike [38] used the Hankel transformation method to solve the closed-form Westergaard half-space problem for concentrated and line load. He distributed real loads acting on the surface of the Westergaard half-space. Ike [38] found that the Hankel transform method is ideal for solving the problem because of the axisymmetric nature of the vertical point load at the origin of Westergaard half-space problem and the simplification offered by the Hankel transformation for axisymmetric problems. The point load solution was then used as the integral kernel in [38] to develop line solutions and distributed real loads. The stress solutions derived by Ike [38] were identical with previous stress solutions derived by Westergaard and others.

Ike [40] used the Mellin transform method to obtain closed-form stress solutions for two-dimensional elasticity problems formulated in-plane polar coordinates. Ike [41] used the cosine integral transformation method to obtain closed-form expressions for stresses caused by point load on the Westergaard half-space. Ike [42] used the Fourier cosine transform method to solve the 2D elasticity problem of point load on an elastic half-plane. Ike [43] used the exponential Fourier transform method to obtain stress solutions for elastic half-plane problems. Ike [44] derived closed-form solutions to Navier's equations for axisymmetric elasticity problems of the elastic half-space. Ike et al. [45] solved elastic half-space problems using Trefftz displacement potential function method. Ike et al. [46] used Bessel function and potential method to solve the axisymmetric elasticity problem of elastic half-space. Onah et al. [47] used the Boussinesq displacement function method to derive vertical stresses and displacement fields due to distributed loads on elastic half-space.

Gibson [48] studied the non-homogeneous isotropic elastic half-space problem subjected to loading normal to its plane boundary. In the work, the non-homogeneity was considered by assuming that Young's modulus of the half-space varied with depth. The researcher in [48] presented a detailed study of an incompressible elastic half-space with Young's modulus E(z) increasing linearly with depth, z.

Cawler and Christian [49] developed and implemented computer codes/programs for the finite element analysis of non-homogeneous elastic half-space problems under circular foundation loadings. They fixed that their computer program results were identical to theoretical results obtained from straightforward applications of theoretical answers in the literature.

Te-Martirosyan [50], Flovin [51], Sitharam and Govindaraju [52] and Taylor [53] have studied stresses elastic half-space problems of the Boussinesq and Westergaard types. Taylor [53] extended the fundamental (point load) solution for the Westergaard elastic half-space under point load to find the vertical stresses due to uniformly distributed load over a rectangular foundation using the superposition method and evaluating the resulting multiple integration problems.

Bhushan and Haley [54] presented novel solutions for stress fields in embedded foundations in a Westergaard half-space, which were not previously available. They also compared settlements based on Boussinesq, Mindlin, Westergaard, and Mindlin-Westergaard (embedded Westergaard) half-space vertical stress distributions. Sadek and Shahrour [55] used finite element modeling to investigate the influence of elastoplasticity on the Boussinesq solutions for stresses in an elastic half-space. In their work, they compared the Boussinesq elastic stress distribution with the elastic stresses from elastoplastic finite element analysis and found that plasticity reduces the attenuation of the critical stresses in the soil mass, implying that the elastic stresses from Boussinesq theory is an underestimation for the areas contributing to soil settlements.

Ike [56] used Love stress function method to solve axisymmetric elasticity problems of the elastic half-space. He obtained general expressions for stresses and displacements due to point and uniformly distributed loadings on the Boussinesq elastic half-space. The paper aims at solving Westergaard's half-space problems using potential theory. The objectives include to:

1) Formulate the governing Cauchy-Navier equilibrium equation for the Westergaard problem.
2) Express the displacement formulation of the equilibrium of the Westergaard half-space as a three-dimensional Laplacian problem in Cartesian Coordinates.
3) Obtain the general solution for stresses and displacements in the Westergaard half-space subject to vertical point load applied at a reference point on the surface by using potential theory.

2. Theoretical Framework

2.1 Westergaard’s Theory Assumptions

The assumptions of the Westergaard’s theory are as follows: [35, 38, 41].

The elastic (soil) medium is semi-infinite in extent. Still, it contains many close spaces horizontal sheets with an insignificant thickness of an infinite rigid material that allows only vertical strain and restrains lateral strain.

Westergaard’s assumptions are close to practical reality, particularly for overconsolidated and laminated sedimentary or stratified soils showing considerable anisotropy.

Westergaard’s assumptions are close to practical reality, particularly for overconsolidated and laminated sedimentary or stratified soils showing considerable anisotropy. Westergaard’s stress analysis in soil continuum under loaded foundation areas applies to stratified soils.

2.2 Framework of The Theory of Elastic Half-Space Problems

Elastic half-space problems simultaneously satisfy the kinematic equations, generalized Hooke’s stress-strain relations, and the differential equations of equilibrium [4-6], [57-59].

2.2.1 Kinematic equations

The kinematic equations for small displacement linear elasticity are six equations relating strains to displacements [8-10], [60-62].

2.2.2 Constitutive equations

The generalized Hooke’s stress–strain relations are six equations [44-46,63].

2.2.3 Differential equations of equilibrium

The differential equations of equilibrium for elastostatic problems where body forces are disregarded are a set of three equations [45-47,60,63].

2.2.4 Navier-Lame displacement equations of equilibrium

2.2.4.1 Stress–displacement equations

The stress–displacement equations obtained by substituting the strain–displacement relations in the stress-strain equations are:

$$\sigma_{xx} = \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2G\frac{\partial u_x}{\partial x}$$

$$\sigma_{yy} = \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2G\frac{\partial u_y}{\partial y}$$

$$\sigma_{zz} = \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2G\frac{\partial u_z}{\partial z}$$

$$\tau_{xy} = G\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$\tau_{yz} = G\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)$$

$$\tau_{xz} = G\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \quad (1)$$

where $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$are normal strains in the x, y and z coordinate directions respectively, y_{xy}, y_{yz}, y_{zx} are shear strains, ux, uy and $u_z$are the displacement field components in the x, y and z coordinate directions.

$$G = \frac{E}{2(1+\mu)} \tag{2}$$

$$\lambda = \frac{2\mu G}{1-2\mu} \tag{3}$$

ε_v is the volumetric strain, G is the shear modulus, E is the Young's modulus and μ is the Poisson's ratio. σ_{xx}, σ_{yy}, σ_{zz} are the normal stresses; τ_{xy}, τ_{xy} and τ_{xz} are the shear stresses. λ is Lamés constant.

The Navier-Lame displacement equations of equilibrium obtained by substituting the stress–displacement equations in the differential equation of equilibrium give the three sets of partial differential equations. For equilibrium in the x direction,

$$\frac{\partial}{\partial x}\left(\lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2G\frac{\partial u_x}{\partial x}\right) + \frac{\partial}{\partial y}G\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) + \frac{\partial}{\partial z}G\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = 0 \tag{4}$$

Simplifying gives:

$$G\nabla^2 u_x + (\lambda + G)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) = 0 \tag{5}$$

$$\text{Or, } \nabla^2 u_x + \left(\frac{\lambda + G}{G}\right)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) = 0 \tag{6}$$

where ∇^2 is the Laplacian.

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{7}$$

Similarly, for equilibrium in the y direction,

$$\nabla^2 u_y + \left(\frac{\lambda + G}{G}\right)\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) = 0 \tag{8}$$

For equilibrium in the z-direction,

$$\nabla^2 u_z + \left(\frac{\lambda + G}{G}\right)\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) = 0 \tag{9}$$

2.3 Governing Partial Differential Equations of Equilibrium For Westergaard Problems – Displacement Formulation

The governing partial differential equations (PDEs) of equilibrium for Westergaard problems can be obtained in a displacement formulation from the simplification of the Navier-Lamé PDEs. For Westergaard problems, the assumptions of horizontal inextensibility results in:

$$u_x = u_y = 0$$
$$\text{And, } u_z \neq 0 \tag{10}$$

Then, the Navier-Lamé PDEs reduce to:

$$\frac{\lambda + G}{G}\frac{\partial}{\partial x}\frac{\partial u_z}{\partial z} = 0 \tag{11}$$

$$\therefore \quad \frac{\partial^2 u_z}{\partial x \partial z} = 0 \tag{12}$$

$$\text{Since, } \frac{\lambda + G}{G} \neq 0 \tag{13}$$

$$\therefore \quad \frac{\partial^2 u_z}{\partial y \partial z} = 0 \tag{14}$$

Equilibrium in the *z* direction gives:

$$\nabla^2 u_z + \frac{\lambda + G}{G}\frac{\partial}{\partial z}\frac{\partial u_z}{\partial z} = 0 \tag{15}$$

$$\nabla^2 u_z + \frac{\lambda + G}{G}\frac{\partial^2 u_z}{\partial z^2} = 0 \tag{16}$$

Expanding, we have

$$\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2} + \frac{\lambda + G}{G}\frac{\partial^2 u_z}{\partial z^2} = 0 \tag{17}$$

$$\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \left(1 + \frac{\lambda + G}{G}\right)\frac{\partial^2 u_z}{\partial z^2} = 0 \tag{18}$$

$$\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \left(\frac{\lambda + 2G}{G}\right)\frac{\partial^2 u_z}{\partial z^2} = 0 \tag{19}$$

$$\text{Let } \frac{\lambda + 2G}{G} = \frac{1}{\beta^2(\mu)} \tag{20}$$

$$\text{Then, } \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{1}{\beta^2(\mu)}\frac{\partial^2 u_z}{\partial z^2} = 0 \tag{21}$$

$$\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial(\beta(\mu)z)^2} = 0 \tag{22}$$

$$\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z_1^2} = 0 \tag{23}$$

$$\text{Where, } z_1 = \beta(\mu) z \tag{24}$$

$$\nabla^2 u_z(x, y, z_1) = \nabla^2 u_z(x, y, \beta(\mu)z) = 0 \tag{25}$$

The governing PDE for equilibrium in the z direction is thus a 3D Laplacian equation in x, y, and $\beta(\mu)z$ and can thus be solved using potential function methods [64-66].

3. Solution by Potential Function Method

The general solution for u_z is a potential function in the x, y, βz space given as [64], [65], [66]:

$$u_z = c_1 \frac{1}{R} = \frac{c_1}{(x^2 + y^2 + (\beta(\mu)z)^2)^{1/2}} \tag{26}$$

$$\text{Wherein, } R^2 = x^2 + y^2 + (\beta(\mu)z)^2 \tag{27}$$

The symbol, c1 is integration constant.
Proof that $u_z = c_1(x^2 + y^2 + (\beta(\mu)z)^2)^{-1/2}$ is a solution to Equation (29).
By partial differentiations,

$$\frac{\partial^2 u_z}{\partial x^2} = -c_1 \left\{ \frac{1}{R^3} - \frac{3x^2}{R^5} \right\} \tag{28}$$

$$\frac{\partial^2 u_z}{\partial y^2} = -c_1 \left\{ \frac{1}{R^3} - \frac{3y^2}{R^5} \right\} \tag{29}$$

$$\frac{\partial^2 u_z}{\partial z_1^2} = -c_1 \left\{ \frac{1}{R^3} - \frac{3z_1^2}{R^5} \right\} \tag{30}$$

Then,

$$\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z_1^2} = -c_1 \left\{ \frac{1}{R^3} + \frac{1}{R^3} + \frac{1}{R^3} - 3\left(\frac{x^2 + y^2 + z_1^2}{R^5} \right) \right\} = -c_1 \left(\frac{3}{R^3} - \frac{3R^2}{R^2} \right) = 0 \tag{31}$$

The obtained general solution to uz is valid at all points in the infinite half-space except at the origin, where $u_z = \infty$ and the problem becomes indeterminate. The general solution for σzz is obtained using the stress displacement relations.

$$\sigma_{zz} = G\beta^{-2} \frac{\partial}{\partial z} u_z = G\beta^{-2} c_1 \frac{\partial}{\partial z} (x^2 + y^2 + \beta^2(\mu) z^2)^{-1/2}$$
$$= -Gc_1 z (x^2 + y^2 + \beta^2(\mu) z^2)^{-3/2} \tag{32}$$

$$\sigma_{xx} = \sigma_{yy} = \frac{\mu}{1-\mu}\sigma_{zz} \quad (33)$$

$$\tau_{xy} = \tau_{yx} = 0 \quad (34a)$$

$$\tau_{yz} = \tau_{zy} = -Gc_1 y(x^2 + y^2 + \beta^2(\mu)z^2)^{-3/2} \quad (34b)$$

$$\tau_{xz} = \tau_{zx} = -Gc_1 x(x^2 + y^2 + \beta^2(\mu)z^2)^{-3/2} \quad (34c)$$

4. Results

4.1 Point Load at The Origin of The Westergaard Half-Space

A point load Q_0 applied at the origin O on a Westergaard half-space shown in Figure 1 is considered.

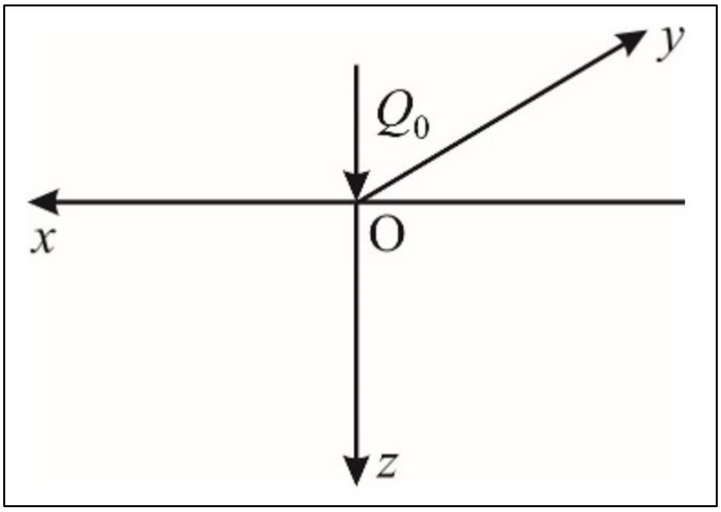

Figure 1: Point load Q_0 applied at the origin O on a Westergaard half-space

The equilibrium of the applied vertical load and the vertical stresses is used to obtain the integration constant c_1. Hence, for an equilibrium of vertical load and the vertical stresses, we have:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma_{zz}(x, y, z)dxdy + Q_0 = 0 \quad (35)$$

$$Q_0 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma_{zz}(x, y, z)dxdy \quad (36)$$

By coordinate transformation from Cartesian to cylindrical polar coordinates,

$$Q_0 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma_{zz}(r, z)|J|\, dr d\theta \quad (37)$$

where |J| is the Jacobian of the transformation and $0 \le r \le \infty, 0 \le \theta \le 2\pi, 0 \le z \le \infty$
r is the radial coordinate, θ is the angular coordinate, and z is the depth coordinate.
The relations between 3D Cartesian coordinates (x, y, z) and cylindrical coordinates (r, θ, z) are

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z \tag{38}$$

The Jacobi matrix or Jacobian |J| of the coordinate transformation $(x, y, z) \rightarrow (r, \theta, z)$ is:

$$|J| = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} \tag{39a}$$

$$|J| = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \tag{39b}$$

$$|J| = r \tag{39c}$$

$$\text{and } dV = dxdydz = |J|\, dr d\theta dz \tag{39d}$$

$$dV = r\, dr\, d\theta dz \tag{39e}$$

$$Q_0 = -\int_0^{2\pi}\int_0^{\infty} (-Gc_1 z)(r^2 + \beta^2(\mu)z^2)rdrd\theta \tag{40}$$

$$Q_0 = \frac{2\pi G c_1}{\beta(\mu)} \tag{40a}$$

$$c_1 = \frac{Q_0\beta(\mu)}{2\pi G} \tag{41}$$

Hence,

$$u_z = \frac{Q_0\beta(\mu)}{2\pi G}(x^2 + y^2 + (\beta^2(\mu)z^2))^{-1/2} \tag{42}$$

Stresses:

$$\sigma_{zz} = \frac{-GQ_0\beta(\mu)z}{2\pi G}(x^2 + y^2 + \beta^2(\mu)z^2)^{-3/2} =$$

$$\sigma_{zz} = \frac{-Q_0\beta(\mu)z}{2\pi}(x^2 + y^2 + \beta^2(\mu)z^2)^{-3/2} \tag{43}$$

$$\sigma_{zz}(r,z) = \frac{-Q_0\beta(\mu)z}{2\pi}(r^2+\beta^2(\mu)z^2)^{-3/2} \tag{44}$$

$$\sigma_{zz} = \frac{-Q_0\beta(\mu)}{2\pi z^2}\left(\frac{z^3}{(r^2+\beta^2(\mu)z^2)^{3/2}}\right) \tag{45}$$

$$\sigma_{zz} = \frac{-Q_0}{z^2}\frac{\beta(\mu)}{2\pi}\left(\frac{(z^2)^{3/2}}{(r^2+\beta^2(\mu)z^2)^{3/2}}\right) \tag{46}$$

$$\sigma_{zz} = \frac{-Q_0}{z^2}\frac{\beta(\mu)}{2\pi}\left(\frac{r^2}{z^2}+\beta^2(\mu)\right)^{-3/2} \tag{47}$$

$$\sigma_{zz} = \frac{-Q_0}{z^2}I_w \tag{48}$$

$$I_w = \frac{\beta(\mu)}{2\pi}\left(\beta^2(\mu)+\frac{r^2}{z^2}\right)^{-3/2} = \frac{\beta(\mu)}{2\pi}\left(\beta^2(\mu)+\frac{x^2+y^2}{z^2}\right)^{-3/2} \tag{49}$$

I_w is the Westergaard vertical stress influence coefficient.

Also, σ_{xx} and σ_{yy} are found in Equation (33), and τ_{xy}, τ_{yx} are found in Equation (34a). The shear stresses τ_{yz} and τ_{xz} are found using Equations (34b) and (34c).

$$\tau_{yz} = \tau_{zy} = G\frac{\partial u_z}{\partial y} = G\frac{Q_0\beta(\mu)}{2\pi G}\frac{\partial}{\partial y}(x^2+y^2+\beta^2(\mu)z^2)^{-1/2} \tag{50}$$

$$\tau_{zy} = \frac{-Q_0\beta(\mu)y}{2\pi}(x^2+y^2+\beta^2(\mu)z^2)^{-3/2} \tag{51}$$

$$\tau_{xz} = G\frac{\partial u_z}{\partial x} = \frac{GQ_0\beta(\mu)z}{2\pi G}\frac{\partial}{\partial x}(x^2+y^2+\beta^2(\mu)z^2)^{-1/2} \tag{52}$$

$$\tau_{xz} = \frac{-Q_0\beta(\mu)x}{2\pi}(x^2+y^2+\beta^2(\mu)z^2)^{-3/2} \tag{53}$$

$$\text{For } \mu = 0,\ \ \beta^2(\mu=0) = 0.50 \tag{54}$$

$$I_w(\mu=0) = \frac{\sqrt{0.5}}{2\pi}\left(0.5+\frac{r^2}{z^2}\right)^{-3/2} \tag{55}$$

Table 1: Values of Westergaard vertical stress influence coefficient I_w for various values of r/z for μ = 0 (r is the radial distance from the point load, z is the depth)

r/z	I_w (Present work)	I_w [23, 35, 38, 41]
0	0.3183	0.3183
0.1	0.3090	0.3090
0.2	0.2836	0.2836
0.3	0.2483	0.2483
0.4	0.2099	0.2099
0.5	0.1733	0.1733
0.6	0.1411	0.1411
0.7	0.1143	0.1143
0.8	0.0925	0.0925
0.9	0.0751	0.0751
1.0	0.0613	0.0613
1.10	0.0503	0.0503
1.20	0.0417	0.0417
1.30	0.0347	0.0347
1.40	0.0292	0.0292
1.50	0.0247	0.0247
1.60	0.0210	0.0210
1.70	0.0180	0.0180
1.80	0.0156	0.0156
1.90	0.0135	0.0135
2.0	0.0118	0.0118
3.0	0.0038	0.0038
4.0	0.0017	0.0017
5	0.0009	0.0009
6	0.0005	0.0005
10	0.0001	0.0001
∞	0	0

5. Discussion

This paper presents the Westergaard problem in the theory of elasticity as a 3D Laplacian equation in transformed transverse (depth) coordinates using a displacement formulation. The formulation was accomplished by simultaneously considering the horizontal inextensibility assumptions of the Westergaard problem in the generalized constitutive relations, the small displacement kinematic equations, and the differential equations of equilibrium. The resulting Navier-Cauchy equilibrium equations in displacement terms yield the Laplacian problem in transformed depth coordinates. The theory of potential functions is then used to obtain the general solutions to the Westergaard problem. The general solutions obtained are the displacements and stresses at any point in a Westergaard half-space. They are expressed in terms of an integration constant c_1 as Equations (26), (32), (33), (34a), (34b) and (34c).

The classical Westergaard problem of a point load acting at the origin of the half-space was considered. The equilibrium of internal vertical stresses and external load was used to obtain the previously unknown integration constant as Equation (40). Then the displacements and stresses become Equations (41), (47), (51), (53), (33) and (34a).

Vertical stresses are the most significant of the stresses due to their role in producing elastic displacements and settlements. The vertical stresses are presented using dimensionless factors in Table 1 and compared with previous results by Ike [38] obtained using Hankel transformation method. Ike [41] obtained using the cosine integral transformation method, Bowles [23] and Westergaard [35]. Table 1 shows that the present results are identical to those obtained using the Hankel transformation method.

6. Conclusion

This study uses the potential function method to derive the solutions for stresses in a Westergaard half-space due to a vertical point load acting at the origin of the Cartesian coordinate.

This paper has presented the solutions to Westergaard's half-space problems of the theory of elasticity using potential theory. The considered problem has significant applications in soil mechanics where it is applied to calculate vertical stresses in soil continuum due to applied structural load considered to be acting at a point on the soil boundary. The conclusions of the study are as follows:

1) The simultaneous consideration of the small-displacement strain–displacement relations, the generalized stress–strain laws, and the differential equations of equilibrium together with the horizontal inextensibility assumption result in a displacement formulation of the Westergaard problem.
2) The resulting displacement formulation is a 3D potential problem in transformed z coordinates governed by 3D Laplacian in terms of the x, y, and transformed z coordinates.
3) The problem is then amenable to potential methods of solving 3D Laplacian equations.
4) The general solution for the 3D Laplacian equation is obtained in terms of an integration constant that can be determined for specific Westergaard problems.

5) The general solution is obtained for the displacements and stresses at any point in the Westergaard half-space.
6) The specific problem of point load acting at the origin on the Westergaard half-space is solved by using the requirement of equilibrium of internal vertical stresses and the applied point load to obtain the integration constant.
7) The vertical displacements and the stresses obtained are identical to previously obtained results presented using the Hankel transforms method and the cosine integral transform method.
8) The solutions for the vertical stress field are conveniently expressed in terms of vertical stress influence factors, tabulated in Table 1. Table 1 shows the values of I_w obtained in the study are identical to previous results by using Hankel transform method and by using the cosine integral transform method.
9) The expressions obtained for the vertical displacements and stresses are exact within the framework of the theory deployed since they satisfy the governing equations of the problem.
10) The solutions are unbounded at the origin due to the singularities in the expressions for the vertical displacement.

Notations

x, y, z	Cartesian coordinates in three-dimensional geometry
R,r, θ, z	Cylindrical polar coordinates
2D	two-dimensional
3D	three-dimensional
∞	infinity
u_x, u_y, u_z	displacement components in the x, y, and z coordinate directions, respectively.
$\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$	normal strains in the x, y, and z coordinate directions, respectively.
ε_v	volumetric strain
y_{xy}, y_{yz}, y_{zx}	shear strains
μ	Poisson's ratio
λ	Lamés constant
G	shear modulus
E	Young's modulus
$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$	normal stresses
$\tau_{xy}, \tau_{yz}, \tau_{xz}$	shear stresses
∇^2	Laplacian operator
PDE	Partial Differential Equation
PDEs	Partial Differential Equations
$\beta(\mu)$	Parameter defined in terms of μ
$Z1$	transformed z coordinate, defined in terms of $\beta(\mu)$ and z.
$C1$	integration constant
R	radial coordinate in space defined in terms of x, y, and $\beta(\mu)$ z
$Q0$	point load acting on the Westergaard half-space
O	origin of Westergaard half-space
$\|J\|$	Jacobian of coordinate transformation from 3D Cartesian coordinates to cylindrical polar coordinates
$\int$	integration
I_w	Westergaard vertical stress influence coefficient
$\frac{\partial}{\partial x}$	partial derivative with respect to x
r	radial coordinate
$\| \quad \|$	determinant
$\iint$	double integral
z	depth coordinate
θ	angular coordinate
r	radial coordinate

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Data availability statement

The data that support the findings of this study are available on request from the corresponding author.

Conflicts of interest

The authors declare that there is no conflict of interest.

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