Stiffness Matrix for Haunched Members With Including Effect of Transverse Shear Deformations

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Abstract
This study includes the derivation of the stiffness matrix for a haunched member using the simple bending theory. The derived stiffness matrix covers most possible geometric shapes for haunched members under different loading cases and combinations with including transverse shear deformations effect. The importance of the transverse shear deformation in haunched members with high depth to span ratios is shown using numerical example. The accuracy of the proposed analysis technique is verified by comparing the results of the numerical example with those obtained from the general analysis program SAP90 using a large number of subelements.

Introduction
Haunched members can be used to shape the members in accordance with the distribution of the internal stress. By using these types of members, one can achieve the required strength with the minimum weight and material and also may satisfy architectural or functional requirements. In industrial buildings, bridges, and high rise buildings, non-prismatic members with variable depth or width are usually used. Different approaches have been developed for the analysis of non-prismatic members (including haunched members). Reynolds and Steedman (1988) published tables and graphs for analysis purposes. Similar calculations are also given in other textbooks, for example by Timoshenko and Young (1965), Vanderbilt (1978) and Funk and Wang (1988) calculated the stiffness matrix and fixed end forces by dividing the non-prismatic member into subelements.

A refined analysis can be performed by deriving the stiffness matrix and fixed end forces by...
considering the exact variations of the geometry. Al-Gahtani (1996) derived the stiffness matrix by using differential equations and determined fixed end forces for distributed and concentrated member loads. Timoshenko and Young (1965) concluded that if the variation of the cross section of a non-prismatic member is not too rapid, it can be analyzed with sufficient accuracy by using the prismatic beam equations. Al-Mezani and Balkaya (1991) demonstrated the problems due to discontinuity of member axis and arching effect in analyzing non-prismatic members.

The use of tables and graphs is limited to certain cross sections and span loads and is also difficult if several loading may be considered. In the case of computer applications, dividing the members into subelements increases the number of equations and requires a larger amount of input data. Therefore, most of latest studies have focused on a stiffness formulation of non-prismatic members, which considered the exact variation of the geometry. The effect of shear deformations on fixed end forces has not been considered so far for non-prismatic members.

The purpose of this study is to develop an exact solution using the simple bending theory for non-prismatic members with a wide range of span load variations and finite element formulation.

**Stiffness Matrix:**

The haunched members with a rectangular cross section and length $L$ as shown in Fig. 1 is assumed to be made of homogeneous, isotropic and linearly elastic materials. Stiffness Matrix:

The haunched members with a rectangular cross section and length $L$ as shown in Fig. 1 is assumed to be made of homogeneous, isotropic and linearly elastic materials. Member end displacements, $U$, forces $F$, and fixed end forces $P$ are shown in Fig. 2.

The displacement method yields the following member equilibrium equation:

$$F = K \cdot U + P \quad (1)$$

where $K$ is the general stiffness matrix of a non-prismatic member. Based on the conjugate beam method (Norriss et. al. 1982), $K$ can be suggested to be as follows:

\[
\begin{bmatrix}
\frac{n_i E A_0}{L} & 0 & 0 & -n_i E A_0 \frac{L}{L} & 0 & 0 \\
0 & (m_j + m_j + 2m_j) \frac{E I_0}{L} & (m_i + m_j) \frac{E I_0}{L} & 0 & -(m_i + m_j + 2m_j) \frac{E I_0}{L} & (m_j + m_j) \frac{E I_0}{L} \\
0 & (m_i + m_j) \frac{E I_0}{L} & m_i \frac{E I_0}{L} & \Gamma_1 & 0 & -(m_i + m_j) \frac{E I_0}{L} & m_i \frac{E I_0}{L} \\
-n_i E A_0 \frac{L}{L} & 0 & 0 & n_i E A_0 \frac{L}{L} & 0 & 0 \\
0 & -(m_i + m_j + 2m_j) \frac{E I_0}{L} & -(m_i + m_j) \frac{E I_0}{L} & 0 & (m_i + m_j + 2m_j) \frac{E I_0}{L} & -(m_j + m_j) \frac{E I_0}{L} \\
0 & (m_j + m_j) \frac{E I_0}{L} & m_j \frac{E I_0}{L} & \Gamma_2 & 0 & -(m_j + m_j) \frac{E I_0}{L} & m_j \frac{E I_0}{L}
\end{bmatrix}
\]
where \( A_0 \) and \( I_0 \) are the area and moment of inertia for the prismatic part of haunched member respectively and \( E \) is the modulus of elasticity.

The effect of the variation of the area is expressed by the coefficients \( n_{ii} \) and the variation of the moment of inertia by the coefficients \( m_{ii}, m_{jj}, \) and \( m_{ij}. \) When the member is prismatic, \( n_{ii} = 1, m_{ii} = 4, m_{jj} = 2. \) The factors \( \beta, \Gamma_1, \Gamma_2, \) and \( \Gamma_3 \) account for the shear effect. In the case of Bernoulli-Euler theory, \( \beta = \Gamma_1 = \Gamma_2 = \Gamma_3 = 1 \) (no transverse shear effect). However, especially for members with high depth to span ratios, shear deformations should be considered to increase the accuracy.

The coefficients \( m_{ii}, m_{jj}, \) and \( m_{ij} \) are determined by using the conjugate beam method and \( n_{ii} \) is derived from the force-deformation. Fig. 3a shows the corresponding forces and couples when \( P = 0, U_3 = 1, \) and all other displacements are zero in eq. (1). 

\( M(x)/EI(x) \) is applied as load to the conjugate beam as shown in Fig. 3d. From the equilibrium at the \( i \) and \( j \) ends, the following equations are obtained:

\[
\begin{align*}
-AL - m_{ii} \int_0^L \frac{(L-x)^2}{EI(x)} dx + m_{ij} \int_0^L \frac{x(L-x)}{EI(x)} dx &= 0 \\
BL + m_{ii} \int_0^L \frac{x(L-x)}{EI(x)} dx - m_{ij} \int_0^L \frac{x^2}{EI(x)} dx &= 0
\end{align*}
\]

Denoting the integral as:

\[
\begin{align*}
L_1 &= \int_0^L \frac{x^2}{EI(x)} dx, \quad L_2 = \int_0^L \frac{1}{EI(x)} dx, \quad I_1 = \int_0^L \frac{x}{EI(x)} dx \\
I_2 &= \int_0^L \frac{1}{EI(x)} dx
\end{align*}
\]

then equations (3) and (4) can be rewritten in the form:

\[
\begin{pmatrix}
A \\
B
\end{pmatrix} = \begin{pmatrix}
I_0 & -L^2 I_2 + 2LI_3 - I_1 & LI_3 - I_1 \\
L^3 & -LI_3 + I_1 & I_1
\end{pmatrix}
\begin{pmatrix}
m_{ii} \\
m_{ij}
\end{pmatrix}
\]

\[
\begin{pmatrix}
m_{ii} \\
m_{ij}
\end{pmatrix} = \begin{pmatrix}
-1 \\
0
\end{pmatrix}
\]

This yields:

\[
\begin{align*}
m_{ii} &= \frac{I_1}{\text{det} C} \\
m_{ij} &= \frac{LI_3 - I_1}{\text{det} C} \\
m_{ji} &= \frac{-L^2 I_2 + 2LI_3 - I_1}{\text{det} C}
\end{align*}
\]

where \( \text{det} C = -\frac{I_0(I_1 I_2 - L^2)}{L^2} \) when \( P=0, \) and \( U_1 = 1 \) and all other displacements are zero in eq. (1), an axial force of magnitude \( n_{ii} EA/L \) is developed. Then the force-deformation relation gives:
\[
\Delta L = n_i = \frac{EA_0}{L} \int_0^L \frac{1}{A(x)} dx \quad (10)
\]

Solving eq. (10) for the unknown \( n_i \) yields:

\[
n_i = \frac{L}{A_0} \int_0^L \frac{1}{A(x)} dx = \frac{L}{A_0 I_4} \quad (11)
\]

where

\[
I_4 = \int_0^L \frac{1}{A(x)} dx \quad (12)
\]

Determination of the factors \( n_i \), \( m_i \), and \( m_j \) of the basic stiffness coefficients is based on the evaluation of the integrals of eqs. (5) and (12). The integrals \( I_1 \), \( I_2 \), \( I_3 \), and \( I_4 \) for the selected member types can be calculated for different haunch shapes according to the taper factors \( \alpha_i \) and \( \alpha_j \) at ends (see Appendix (A)). A hunched member is assumed to be consisting of three segments as shown in Fig. 1. The moment of inertia and the area in the regions with length \( L_i \) and \( L_j \) are variable and in the region with length \( L_o \) constant. The integrals \( I_1 \), \( I_2 \), \( I_3 \), and \( I_4 \) represent the summation of these three components. Integrals for the constant and linear variations can be performed analytically, while for any curved variation, numerical integration is suggested to be done.

**Stiffness Factors for Shear Effect:**

Neglecting axial deformations and partitioning the matrices into four 2 x 2 submatrices, the beam stiffness matrices can be represented respectively in the following simplified forms:

\[
\begin{bmatrix}
\bar{k}_{11} & \bar{k}_{12} \\
\bar{k}_{21} & \bar{k}_{22}
\end{bmatrix}, \quad \bar{\xi} = \begin{bmatrix}
\bar{\xi}_{11} \\
\bar{\xi}_{21}
\end{bmatrix}
\quad (13)
\]

The stiffness matrix of the member shown in Fig. 4 corresponding to the coordinates \( F_2 \) and \( F_3 \) is:

\[
\bar{k}_{11} = \begin{bmatrix}
\frac{(m_i + m_j + 2m_o)EI_0}{L^2} & \frac{(m_i + m_j)EI_0}{L^2} \\
\frac{(m_i + m_j)EI_0}{L^2} & \frac{m_j EI_0}{L^2}
\end{bmatrix} \quad (14)
\]

Using the relation between the transverse shearing deformations and applied shearing forces, shear deflection due to unit transverse load at end \( i \) can be expressed as Popov (1968):

\[
U_{2_{\text{shear}}} = \frac{\alpha}{G} \int_0^L \frac{1}{A(x)} dx = \frac{\alpha I_4}{G} \quad (15)
\]

where \( G \) is the shear modulus and \( \alpha \) is the shape factor of the cross section for shear.

The following notation is used:

\[
\psi = \frac{\alpha I_4}{G} \frac{(m_i m_j - m_o^2)EI_0}{m_j L} \quad (16)
\]

Shear deformation can be added to the inverse of eq. (14), i.e., the flexibility matrix as:

\[
\bar{\xi}_{11}^{-1} = \begin{bmatrix}
\frac{m_i L^3(1+\psi)}{(m_i + m_j)^2} & - \frac{(m_i + m_j)L^2}{(m_i + m_j + m_j)^2} \\
- \frac{(m_i + m_j)^2}{(m_i + m_j)^2} & \frac{(m_i + m_j + m_j)L}{(m_i + m_j + m_j)^2}
\end{bmatrix} \quad (17)
\]
The stiffness matrix $\bar{k}_{ij}$ by considering the transverse shear deformation can be obtained by inverting eq. (17). The matrix $\bar{k}_{ij}$ can be obtained from the equilibrium conditions of the member. Repeating the outlined procedure for the cantilever, by considering joint $i$ as fixed end and releasing joint $j$, $\bar{k}_{2j}$ and $\bar{k}_{12}$ can be obtained similarly. The multiplication factors $\beta$, $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ are given as follows:

$$\beta = \frac{GL^3}{EI_d I_4 (m_i + m_j + 2m_0)} + 1 \beta$$  
$$\Gamma_1 = \left\{ \frac{EI_d I_4 (m_i + m_j + 2m_0)}{m_j L^2 G} + 1 \right\} \beta$$  
$$\Gamma_2 = \left\{ \frac{-EI_d I_4 (m_i + m_j + 2m_0)}{m_j L^2 G} + 1 \right\} \beta$$  
$$\Gamma_3 = \left\{ \frac{EI_d I_4 (m_i + m_j + 2m_0)}{m_j L^2 G} + 1 \right\} \beta$$

**Fixed End Forces Due to Axial Forces:**

Fixed end forces for the axially loaded non-prismatic member are derived using the flexibility method. Referring to Fig. 5, static equilibrium yields: 

$$P_1 + P_2 = -Q$$  

Choosing $P_1$ as a redundant and loading the primary structure with the actual load $Q$, the displacement of point $i$ due to $P_1$ is:

$$\delta_i = \frac{P_1}{n_i A_0 E / L} = \frac{P_1 I_4}{E}$$  

and the displacement of point $i$ due to the actual force $Q$ is:

$$\delta_{eq} = \frac{Q}{\bar{A}_0 A_0 E / L} = \frac{Q I_5}{E}$$  

where $A_0$ is the minimum area along the length $L$,

$$\bar{A}_0 = \frac{L}{A_0}$$

Using compatibility at point $i$ and the equilibrium condition (eq. (22)), then:

$$p_4 = -\frac{Q I_5}{I_4}$$  

and

$$p_4 = Q \left( \frac{I_4}{I_4} - 1 \right)$$

**Fixed End Forces Due to Bending and Shear:**

The rest of the span loads considered are perpendicular to the member axis, therefore $P_1 = P_4 = 0$. The force $P_3$ and $P_6$ are calculated from:

$$\begin{bmatrix} P_3 \\ P_6 \end{bmatrix} = \begin{bmatrix} EI_0 \left[ \begin{array}{cc} m_i & m_j \\ m_j & m_j \end{array} \right] \theta_j \end{bmatrix} \left( \begin{array}{c} \theta_j \\ \theta_j \end{array} \right)$$

where $\theta_i$ and $\theta_j$ are the rotations at the ends of the member due to the span load $Q(\chi)$. Using the conjugate beam method, these rotations, which are also given by Timoshenko and Young (1965), can be expressed as:
\[ \theta_j = -\frac{1}{EL} \int_0^L \frac{\bar{M}(x)(L-x)}{I(x)} \, dx \]
\[ \theta_j = \frac{1}{EL} \int_0^L \frac{\bar{M}(x)x}{I(x)} \, dx \] (29)

where \( \bar{M}(x) \) is the moment function due to span load \( Q(x) \) of the simply supported beam. The numerical integration is used for the calculations of \( \theta_i \) and \( \theta_j \) in eq. (29). Adding rotations for the different loads and substituting into eq. (28) yields the fixed end forces \( P_3 \) and \( P_6 \). Then \( P_2 \) and \( P_5 \) can be calculated by adding the respective simply supported beam and end rotations and the reactions due to the end moments.

Deformations of the cross section due to transverse shear will produce additional deflection which is calculated by using the Bernoulli-Euler beam theory. The non-prismatic member shown in Fig.1 is subjected to member forces \( Q(x) \). Adding shear deformations over the length, the net deflection at end \( j \) with respect to origin \( i \) can be expressed as:

\[ U_{s,\text{shear}} = \frac{\alpha}{G} \int_0^L \mathcal{V}_i(x) + V_2 \frac{A(x)}{A} \, dx \] (30)

where \( \mathcal{V}_i(x) \) is the function of shear force for the simply supported beam, which is obtained by differentiating the \( \bar{M}(x) \) function with respect to \( x \), and \( V_2 \) is the constant shear force due to end moments. From equilibrium,

\[ V_2 = -\frac{E I_j}{L} \left[ (m_i + m_j) \theta_i + (m_j + m_i) \theta_j \right] \] (31)

Therefore, \( U_{s,\text{shear}} \) can be calculated by using numerical integration.

Additional fixed end forces caused by shear effects are the fixed end forces that will prevent \( U_{s,\text{shear}} \) in the beam. If the span load and the non-prismatic member are symmetric, then \( U_{s,\text{shear}} = 0 \) and the additional fixed end forces will also equal to zero. Otherwise, a correction factor due to shear can be determined from eq. (1) and eq. (2) as follows:

\[
\begin{bmatrix}
P_1 \\
P_2 \\
P_3 \\
P_4 \\
P_5,\text{shear}
\end{bmatrix} = \beta \frac{EI}{L} \begin{bmatrix}
(m_i + m_j) + 2m_i \\
-m_i \frac{L}{m_i + m_j} + m_i \\
(m_j + m_i) + 2m_j \\
-m_j \frac{L}{m_j + m_i} + m_j
\end{bmatrix} U_{s,\text{shear}}
\] (32)

Finally, the vector \( P \) is determined by assigning the corresponding components of the fixed end forces due to axial forces, bending and shear.

**Numerical Example**

For the fixed ends haunched beam shown in Fig. 6, the elastic modulus of concrete, \( E \), is taken as 30 GPa, the shear modulus, \( G \), as 12 GPa, the shape factor for shear, \( \alpha \), as 1.2 (for rectangular cross section), and member width is 0.4m.

The beam is analyzed by using the proposed method for mid-span heights, \( h \), of 0.5, 0.75, 1.0, 1.25, and 1.5m. Fixed end moments at the left and right supports are determined with and without the transverse shear deformations. The results are compared with that obtained using the well-known finite element analysis program SAP90. For SAP90 analysis, the beam is divided into 48 subelements having constant area and moment of inertia defined at mid-section of each element with and without considering the effect of transverse shear deformations. The result of analysis is shown in Table...
1A and Table 1B for both typed of analysis. When shear deformation is considered, the end moments increase. It is clear that the effect of shear deformations increases as the depth to span ratio increases.

**Conclusions:**
1-The derived stiffness matrix is general and applicable to simple bending theory, and cover a wide range of depth variation for haunched beams.
2-The proposed formulation is also general and can be used for other types of non-prismatic or haunched members.
3-Fixed end forces due to transverse shear deformations are considered, therefore more accurate results can be obtained in the case of high depth to span ratios.
4-Members with haunches can be analyzed as one element. This will reduce the number of equations, input data, time and effort compared to the analysis method of dividing into prismatic subelements, and that is very useful in frame analysis with non-prismatic members.
5-The proposed element is convenient to use with the general displacement method. It can be adapted to any finite elements analysis program.

**List of Symbols:**
- $a$: subdivision length (Fig. 5)
- $A, B$: left and right support reaction
- $A_i, A_j$: cross sectional area at $i$ and $j$ ends
- $A(x)$: cross section area of the beam
- $A_0$: min. area of the member cross section
- $\overline{A}_0$: min. area along the length $\overline{L}$
- $C$: matrix of the coeff. in eq. (6)
- $E$: modulus of elasticity
- $F$: member force vector
- $F_1, F_6$: member forces
- $G$: shear modulus
- $I_i, I_j$: moment of inertia at $i$ and $j$ ends
- $I(x)$: moment of inertia at $x$ from origin
- $I_o$: min. moment of inertia of member cross section
- $I_{1,1}, I_{3,3}$: integrals given in eqs. (5) and (12)
- $I_5$: integral given in eq. (25b)
- $K_{23, 63}$: stiffness coeff. for $U_{5,1}$
- $K, \overline{K}$: beam stiffness with neglecting axial deformation
- $K$: member stiffness matrix
- $L$: length of member
- $L_i, L_j$: length in variable left and right haunches
- $L_o$: length in the constant mid-section.
- $L$: subdivision length
- $M(x)$: moment function of the beam
- $\overline{M}(x)$: moment function due to span load $M(x)$ of simply supported beam
- $n_i, n_6, n_{ij}, n_k$: factors of the basic stiffness coefficients
- $\overline{n}_{ii}$: factor of the stiffness coeff. for length $\overline{L}$
- $P$: member fixed end
forces vector
\( P_1, P_6 \) fixed end forces
\( q(x) \) addition of \( q_1 \) and \( q_2 \)
\( q_1, q_2 \) \( M(x)/E I(x) \) function due to left right moment
\( Q \) load intensity
\( Q(x) \) span load function
\( U \) member displacement vector
\( U_1, U_6 \) member disp.
\( U_{2\text{shear}} \) shear deflection
\( U_{5\text{shear}} \) total shear deflection of non-prismatic member at right end
\( \nabla_1(x) \) function of shear force for simply supported beam
\( V_2 \) constant shear force due to end moment
\( \alpha \) shape factor for shear
\( \alpha_i \) taper factor at the left and right haunches
\( \beta, \Gamma_1, \Gamma_2, \Gamma_3 \) shear consideration factors
\( \Delta L \) total axial elongation in length \( L \)
\( \delta_Q \) disp. of point \( i \) due to the actual force \( Q \)
\( \delta_{P_i} \) disp. of point \( i \) due to \( P_1 \)

\( \theta_l, \theta_r \) rotations at the left and right supports of the simply support beam
\( \psi \) factor defined in eq. (16)

References:


Appendix A:

Depth Equations for Haunches:

(1) Stepped Haunches:
\[
h(x) = h_i = \text{const. for } 0 \leq x \leq L_i
\]
\[
h(x) = h_o = \text{const. for } L_i \leq x \leq (L_i + L_o)
\]
\[
h(x) = h_o = \text{const. for } (L_i + L_o) \leq x \leq L
\]

(2) Linear Haunches:
\[
h(x) = h_i - \alpha_i x \text{ for } 0 \leq x \leq L_i
\]
\[
h(x) = h_o = \text{const. for } L_i \leq x \leq (L_i + L_o)
\]
\[
h(x) = h_i - \alpha_i (L - x) \text{ for } (L_i + L_o) \leq x \leq L
\]

(3) Parabolic Haunches:
\[
h(x) = \frac{\alpha_i x^2}{L_i} - 2\alpha_i x + h_i \text{ for } 0 \leq x \leq L_i
\]
\[
h(x) = h_i = \text{const. for } L_i \leq x \leq L
\]
\[
h(x) = \alpha_i \left( \frac{x^2}{L_i} - 2(L - L_i) - \frac{(L - L_o)^2}{L_i} + h_o \right) \text{ for } (L_i + L_o) \leq x \leq L
\]

Table 1A: Fixed end moments, kN.m
(Proposed Analysis)

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<tr>
<th>h,m</th>
<th>0.5</th>
<th>0.75</th>
<th>1.0</th>
<th>1.25</th>
<th>1.5</th>
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<td>Left support</td>
<td>With Shear effect</td>
<td>29</td>
<td>5.3</td>
<td>276</td>
<td>.7</td>
</tr>
<tr>
<td></td>
<td>W/O shear effect</td>
<td>29</td>
<td>0.7</td>
<td>269</td>
<td>.1</td>
</tr>
<tr>
<td>Right support</td>
<td>With Shear effect</td>
<td>54</td>
<td>62</td>
<td>67</td>
<td>69</td>
</tr>
<tr>
<td></td>
<td>W/O shear effect</td>
<td>58</td>
<td>69</td>
<td>76</td>
<td>82</td>
</tr>
</tbody>
</table>

Table 1B: Fixed end moments, kN.m
(Finite Element Analysis Using SAP90)

<table>
<thead>
<tr>
<th>h,m</th>
<th>0.5</th>
<th>0.75</th>
<th>1.0</th>
<th>1.25</th>
<th>1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Left support</td>
<td>With Shear effect</td>
<td>295</td>
<td>23</td>
<td>276</td>
<td>91</td>
</tr>
<tr>
<td></td>
<td>W/O shear effect</td>
<td>290</td>
<td>44</td>
<td>269</td>
<td>25</td>
</tr>
<tr>
<td>Right support</td>
<td>With Shear effect</td>
<td>54.8</td>
<td>7</td>
<td>62.6</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>W/O shear effect</td>
<td>58.9</td>
<td>2</td>
<td>69.4</td>
<td>5</td>
</tr>
</tbody>
</table>
Stiffness Matrix for Haunched Members with Including Effect of Transverse Shear Deformations

Fig. 1: Haunched member with variable area and moment of inertia

Fig. 2: (a) member displacements  
(b) member end forces  
(c) fixed end forces

Fig. 3: The C. L. bending and aggregate beam system:
(a) C. L. loading  
(b) Moment diagram for real beam  
(c) M. E. diagram of real beam  
(d) conjugate beam
The image contains diagrams and text related to the stiffness matrix for haunched Members with including effect of transverse shear deformations. The diagrams illustrate non-prismatic members subjected to different forces and loads, including an axially loaded member and a numerical example.

Fig. 4: Non-prismatic member subjected to F2 and F3.

Fig. 5: Axially loaded member.

Fig. 6: Numerical example.