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Finite Element of Linear Viscoelastic Solids

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Abstract

A linear thermoviscoelastic material model, whose basis is an incremental constitutive equation that takes complete strain and temperature histories into account, is derived in finite element code " FEVES "(finite element of viscoelastic solids).

The software " FEVES " can be effectively employed for all permissible values of Poisson's ratio by using a selective integration procedure. The software is tested for a number of real applications and the results are compared with available analytical values.

Keywords: Finite element, thermoviscoelasticity, viscoelastic solids.

الخلاصة

في هذا البحث تم تقديم نموذج رياضي للمسائل المرنة اللزجة الحرارية اعتماداً علــى معادلــة مكوّنة للخصائص حيث تأخذ بنظر الإعتبار الإنفعال الكلي ودرجة الحرارة المسلطة وتمت كتابة برمجيات العناصر المحددة لحل هذا النموذج وسميت باسم"FEVES"

إن هذه البرمجيات تم استخدامها لجميع قيم معامل بوزون الممكنة وبكفائة عاليــة باســتخدام طريقة التكامل الإنتقالي.تم اختبار والتأكد من نتائج هذه البرمجيات من خلال عدد من *التطبيقات العملية حيث قورنت النتائج مع نتائج نظرية منشورة.*

Notation

The following symbols are used in this paper

δ_{ij} = Kronecker delta, $\begin{cases} 1 \text{ when } j = I \\ 0 \text{ when } j = i \end{cases}$

[ ] = matrix

$[K_o]$ = elastic stiffness matrix

$[K_i]$ = viscoelastic stiffness matrix

[N] = matrix of shape functions

{ } = vector

$\{q_i\}$ = nodal displacements vector

$\{x_i\}$ = nodal coordinates vector

A_T = shift factor at time t for reduced time

B = strain-displacement matrix

$\mathbf{D_o}$ = elasticity matrix

$\mathbf{D_i}$ = viscoelasticity matrix

dv = incremental volume of the element

$F_m(t)$ = vector of nodal force due to mechanical loading " pessure, gravity, centrifugal load,... "

$F_T(t)$ = vector of nodal force due to thermal load

G(t) = shear relaxation function

K = bulk modulus

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L {ε} = integral operator
t = time
T(t) = current temperature
$T_s(0)$ = structure reference temperature
α = coefficient of thermal expansion

Introduction

Time-dependent response of both metals and non-metals present analysis problems in many areas of engineering, such as the problem of solid propellant rocket fuels, the problems of turbine blade, and soil mechanics, these require analysis that takes into account viscoelastic response under varying time, temperature and loads.

The finite element techni-que, which has been demons-trated to provide an excellent analysis method for complex geometrical configu-rations for elastic cases, has been extended to provide analysis for linear viscoelastic solids.

The viscoelastic analysis techniques may broadly be classified into three basic approaches, which are Quasi-elastic solutions, Integral transform techniques, and Direct methods.

In the Quasi-elastic solutions [1] the elastic properties equivalent to the corresponding viscoelastic properties at the desired time and temperature are proposed. This approach essentially ignores the entire past history of the load and environment. Therefore this solution yields crude approximation to the true response.

Integral transform technique [2,3,4] is employed by using the elastic solution to obtain the corresponding viscoelastic solution. This approach is exact for which closed form solutions are possible.

The Direct formulations are based on the finite element and boundary element theories, using either the differential form [5] or integral form of the stress-strain relationships [1,6, 7,8].

Most viscoelastic materials are assumed to be incompressible or nearly in-compressible viscoelastic solids. Application of the displacement finite element method for the analysis of such solids yields severally oscillating solution in the stress and strain across the elements. This aspect has been studied for elastic materials and is well documented in literature [9]. This may be overcome by using the following steps :

1- Using a selective integration procedure , which is exact in (3×3) Gauss integration points for the shear components and approxi-mate in (2×2) Gauss integration for the bulk components of the elastic stiffness matrix [10].
2- Using 8-node Serendipity isoparametric element [9].
3- The location of stress and strain output, i.e. the sampling position, within the element can be selected at the (2×2) Gauss points, which are favored and give exact results of stresses and strains. While the results at the geometrical nodes or (3×3) Gauss points may give poor and unreasonable results.

2. Theoretical Analysis

The software "FEVES" is based upon the linear uncoupled thermoviscoelastic formulations and is carried out by using the assumptions that the stress-strain relation is a hereditary integral expression, the reduced time hypothesis is valid, the material is isotropic and homogenous, and the bulk modulus is constant with time.

2.1 Stress-strain relation

The viscoelastic stress-strain relation can be obtained by using the elastic stress-strain relation and employing Laplace and Inverse Laplace transformation [2]. The general thermoelastic stress-strain relation can be written as:

$$\sigma_{ij} = 2G\varepsilon_{ij} + \delta_{ij}(k - \frac{2}{3}G)\varepsilon_{kk} - 3\delta_{ij}\alpha\, k\, \Delta T \quad (1)$$

where G is the shear modulus, k the bulk modulus, α the coefficent of thermal expantion, ΔT the temperature increase and δ_{ij} kronecker delta. Repeated subscribts denote summation for the range of the problem.

Laplace transformation may be applied to the above equation to deduce the following stress component in S-domain.

$$\sigma_{ij} = 2sG(s)\varepsilon_{ij}(s) + \delta_{ij}(k(s) - \frac{2}{3}G(s))s\varepsilon_{kk}(s) - 3\delta_{ij}\alpha\, s\, k(s)\Delta T(s) \quad (2)$$

By Inverse Laplace transformation (convolution integral), the stress variation with the time domain may written as:

$$\sigma_{ij}(t) = 2\int_{0^-}^{t} G(\zeta - \zeta')\frac{\partial \varepsilon_{ij}(t')}{\partial t'}dt' + \delta_{ij}k\varepsilon_{kk}(t') - \frac{2}{3}\delta_{ij}\int_{0^-}^{t} G(\zeta - \zeta')\frac{\partial \varepsilon_{kk}(t')}{\partial t'}dt' - 3\delta_{ij}k\alpha\,\Delta T(t) \quad (3)$$

The discontinuity at t=0, may be eliminated from the above expression to obtain the following:

$$\begin{aligned}\sigma_{ij}(t) &= 2G(\zeta')\varepsilon_{ij}(0) + \int_{0}^{t} G(\zeta - \zeta')\frac{\partial \varepsilon_{ij}(t')}{\partial t'}dt' \\ &+ \delta_{ij}\left[k - \frac{2}{3}G(\zeta - \zeta')\right]\varepsilon_{kk}(0) \\ &- \frac{2}{3}\delta_{ij}\int_{0}^{t} G(\zeta - \zeta')\frac{\partial \varepsilon_{kk}(t')}{\partial t'}dt' - 3\delta_{ij}k\alpha\,\Delta T(t')\end{aligned} \quad (4)$$

Applying integration by parts to the second and third terms, the above equation may be simplified into:

$$\begin{aligned}\sigma_{ij}(t) &= 2G(0)\varepsilon_{ij}(t) - \int_{0^-}^{t} \frac{\partial G(\zeta - \zeta')}{\partial t'}\varepsilon_{ij}(t')dt' \\ &+ \delta_{ij}[k - \frac{2}{3}G(0)]\varepsilon_{kk}(t') \\ &- \frac{2}{3}\delta_{ij}\int_{0^-}^{t} \frac{\partial G(\zeta - \zeta')}{\partial t'}\varepsilon_{kk}(t')dt' - 3\delta_{ij}k\alpha\,\Delta T(t')\end{aligned} \quad (5)$$

where

$$\Delta T(t) = T(t) - T_s(0) \quad (6)$$

The shifted time ζ is related to the real time t, by

$$\zeta = \zeta(t) = \int_{0^-} \frac{dt'}{A_t(T(t))} \quad (7)$$

where A_T is the shift function and is evaluated by using so called WLF equation [1] as:

$$\log A_T = -\frac{C_1\Delta T}{C_2 + \Delta T} = -h(T) \quad (8)$$

or

$$\frac{1}{A_T[T(t)]} = 10^{h(T)} \quad (9)$$

where C_1 and C_2 are material constants.

Equation (5) can be rewritten in matrix form as:

$$\{\sigma\}=[D_o]\{\varepsilon\}+[D_l]\{\varepsilon\} - 3\alpha k(\Delta T(t))\{j\} \quad (10)$$

where

$$D_o=\begin{bmatrix} k+\frac{4}{3}G(0) & k-\frac{2}{3}G(0) & 0 \\ k-\frac{2}{3}G(0) & k+\frac{4}{3}G(0) & 0 \\ 0 & 0 & G(0) \end{bmatrix} \quad (11)$$

$$D_l=\begin{bmatrix} -\frac{4}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & -\frac{4}{3} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad (12)$$

$$L\{\varepsilon(t)\}=\int_{0^+} \frac{G(\zeta-\zeta')}{\partial t'}\{\varepsilon(t')\}dt' \quad (13)$$

$$\{J\}^T=\{1 \quad 1 \quad 0\} \quad (14)$$

2.2 Finite Element Formulation

The displacement and coordinate vectors at any point inside an isoparametric element can be related to the nodal displacements and coordinates by using the shape functions as follow [12]:

$$\{U\}=[N]\{q_i\}= \sum N_i q_i=[N]\{q\} \quad (15)$$

where

$\{x\}=[x_1x_2\ldots]^T$, $\{q\}=[q_1q_2\ \ldots]^T$ and $[N]=[N_1, N_2,\ldots]$

the shape functions

$[N]=[N_1, N_2,\ldots]$ are functions of local coordinates in isoparametric elements.

$\Sigma N_i x_i=[N]\{x\}$

The strain-displacement relation may be written as:

$$\{\varepsilon\}=[B]\{q\} \quad (17)$$

where [B] = [L] [N] and [L] is a matrix of differentiation operators.

The Finite Element Equilibrium equation for the finite element can be written as [12]:

$$\int_v [B]^T \sigma_{ij}(t)dv=F_m(t) \quad (18)$$

where v is the volume domain of the element.

By substituting eq.(5) into eq.(18), it can be proved that:

$$\int_v [B]^T [D_o][B]\{q\}dv+ \int_v [B^T][D_l]L\{\varepsilon(t)\}dv-\{F_T(t)\}=\{F_m(t)\} \quad (19)$$

where:

$$\{F_T(t)\}=3\alpha k\Delta T(t)\int_v [B^T]\{j\}dv \quad (20)$$

Employing the trapezoidal rule for time domain, the second term in eq.(19) can be written for kth time step as:

$$\int_v B^T D_l \, L\{\varepsilon(t)\} dv = \frac{1}{2}[G(0) - G(\zeta_k - \zeta_{k-1})][K_l]\{q(t_k)\}$$
$$+ \frac{1}{2}[G(0) - G(\zeta_k - \zeta_{k-1})][K_l]\{q(t_{k-1})\}$$
$$+[K_l]\left(\sum_{j=1}^{k-2}\left(G(\zeta_k - \zeta_{j+1}) - G(\zeta_k - \zeta_j)\right)\right)\{q^*(t_{j+1})\}$$

(21)

where

$$\{q^*(t_{j+1})\} = \frac{1}{2}[\{q(t_j)\} + \{q(t_{j+1})\}] \quad (22)$$

$$[K_l] = \int_v [B^T][D_l][B] dv \quad (23)$$

The above integration can be performed by finite difference integration as shown in Appendix (A). Substituting eq.(21) into eq.(19), the element equilibrium equation for the finite element is obtained as

$$[K_0] + \frac{1}{2}[G(0) - G(\zeta_k - \zeta_{k-1})]$$
$$[K_l]\{q(t_k)\} =$$
$$\{F_m(t_k)\} + \{F_T(t_k)\} + \{M(t_k)\}$$

(24)

where

$$[K_0] = \int_v [B^T][D_o][B] \; dv \quad (25)$$

where:

$$\{M(t_k)\} = -[K_l]\{\Phi(t_k)\} \quad (26)$$

$$\{\Phi(t_k)\} = \frac{1}{2}\begin{bmatrix} G(0) \\ -G(\zeta_k - \zeta_{k-1}) \end{bmatrix}\{q(t_{k-1})\}$$
$$+ \left(\sum_{j=1}^{k-2}\begin{pmatrix} G(\zeta_k - \zeta_{j+1}) \\ G(\zeta_k - \zeta_j) \end{pmatrix}\right)\{q^*(t_{j+1})\}$$

(27)

Nodal displacements $\{q(t_k)\}$ at kth time step are obtained by solving eq.(24),and the corresponding strain can be evaluated by using eq.(17), finally the stresses are computed as:

$$\{\sigma(t_k)\} = [A_l]\{\varepsilon(t_k)\} + \{A_2\}$$
$$- 3\alpha k \Delta T(t_k)\{q(t_j)\} \quad (28)$$

where

$$[A_l] = [D_o]$$
$$+ \frac{1}{2}[G(0) - G(\zeta_k - \zeta_{k-1})][D_l] \quad (29)$$

$$\{A_2\} = [D_o][B]\{\Phi(t_k)\} \quad (30)$$

$\{M(t_k)\}$ is the memory load vector.

2.3 Incompressibility Consideration

The elastic stiffness matrix $[K_0]$ of eq.(25) may be splite into shear and bulk components [10] as shown below

$$[K_0] = [K_0^v] + [K_0^s] \quad (31)$$

where

$$[K_0^v] = \int_v [B^T][D_o^v][B] dv \quad (32)$$

$$[K_0^s] = \int_v [B^T][D_o^s][B] dv \quad (33)$$

$$[D_o]=\begin{bmatrix} 2G(0) & 0 & 0 \\ 0 & 2G(0) & 0 \\ 0 & 0 & 2G(0) \end{bmatrix} + \left(k-\frac{2}{3}G(0)\right)\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (34)$$

$$[D_o^s]=\begin{bmatrix} 2G(0) & 0 & 0 \\ 0 & 2G(0) & 0 \\ 0 & 0 & 2G(0) \end{bmatrix} \quad (35)$$

$$[D_o^v]=\left(k-\frac{2}{3}G(0)\right)\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (36)$$

The integration of $[K^s_0]$ can be performed by using the standard (3×3) Gauss-Legender formula, while $[K^v_0]$ can be evaluated numerically by using (2×2) Gauss-Legender formula to overcome the singularity due to certain values of Poisson's ratio, when it reaches to 0.5. The above (2×2) Gauss point integration is called the reduced or selective integration and it is recommended only when Poisson's ratio "v" reaches to 0.5, for other values it is found that (3×3) Gauss point integration gives more accurate results.

2.4 Local and Global Smoothing of Stresses and Strains:

The geometrical nodes of the finite element mesh, which are the most useful output locations for stresses and strains, appear to be in the worst sample points for incompressible (or nearly incompressible) materials. It has been shown that the integration points "(2×2) Gauss points" give the best stresses and strains at sample points but the stresses and strain will be discontinuous between the elements. To solve this problem one can use ***local*** and ***global*** smoothing technique [11]. First the smoothing may be performed separately over each individual element and this will be called ***local smoothing***, and then taking the average of stresses and strains at the nodes of all elements meeting at a common node. This will be called ***global smoothing***. The smoothing function is shown in the Appendix (B).

1.Numerical example:

In order to verify the validity of the proposed numerical method, several numerical examples are considered, for which exact and numerical solutions have been already obtained.

3.1 Pressurized Viscoelastic Hollow Cylinder:

A thick viscoelastic hollow cylinder subjected to a constant internal pressure is studied as the first example, the relaxation function of this example are:

$$G(t) = G_0[\beta+(1-\beta)\, e^{-\lambda t}]$$

The constant coefficients of the relaxation function are given in the following table where t_r is the retardation time, which is equal to $(\beta\lambda)^{-1}$.

G_0	K	β	λ	t_r	v
480	1280	0.25	0.4	10	0.3333 (compressible material)

The exact solution for the displacement in r-direction is given by ref. [8]

$$u(r,t)=\frac{Pa^2}{2(b^2-b^2)}\left[\frac{b^2}{Gr\beta}(1-(1-\beta)e^{-\beta\lambda t})+\frac{3r}{(G\beta+3K)}(1-(1-\Phi)e^{-\Phi\lambda t})\right]$$

where P, a and b are the pressure and inner and outer radii respectively as shown in fig.(1).

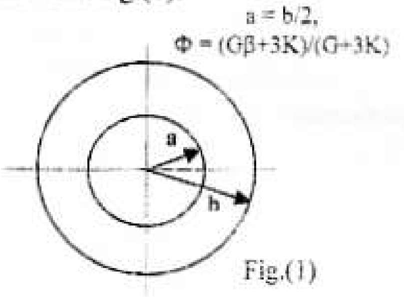


Fig.(1) Thick hollows viscoelastic cylinder under internal pressure

The radial displacement at inner and outer surfaces at different time steps are presented in Fig.(2). It is clear that the "FEVES" code results are very close to the analytical values.

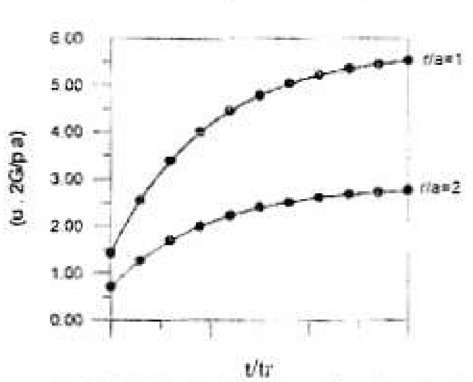


Fig.(2) Radial displacement of a viscoelastic hollow cylinder

3.2 Solid mass slump problem:

Solid propellant grains subjected to gravity force is a serious problem in solid propellant engineering, the material being in nature viscoelastic, the propellant grains stored for along time, undergo dimensional deviations due to their own weight. Normally the grains are supported by a casing. It is expected that the slumping can be minimized by supporting the grain at the bottom. This problem is studied for a simple example of a rectangular prism structure in ref.[5]. Details of the structure, the finite element mesh, and the material properties are shown in Fig. (3):

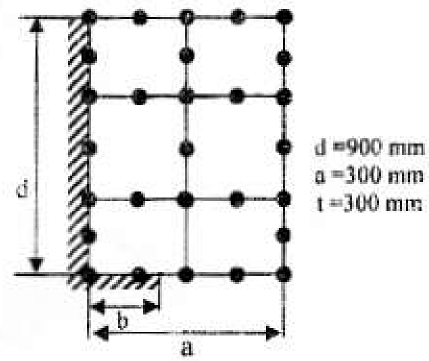


Fig.3.

$G(t)=.022+.03e^{-.0025t}+.048e^{.007t}$

ρ(density) = 1.8 E-6 kg /mm^3 K=110 kg/mm^2

ν (Poisson's ratio) =0.4995456 "*Nearly incompressible*"

This problem is assumed to be nearly incompressible (ν→0.5). The "FEVES" results are shown in fig.(4) (a and b), where the variations of the vertical and horizontal displacements at certain points are given. It is clear that there is a good agreement between the present FEM results and the FEM results from ref.[5].

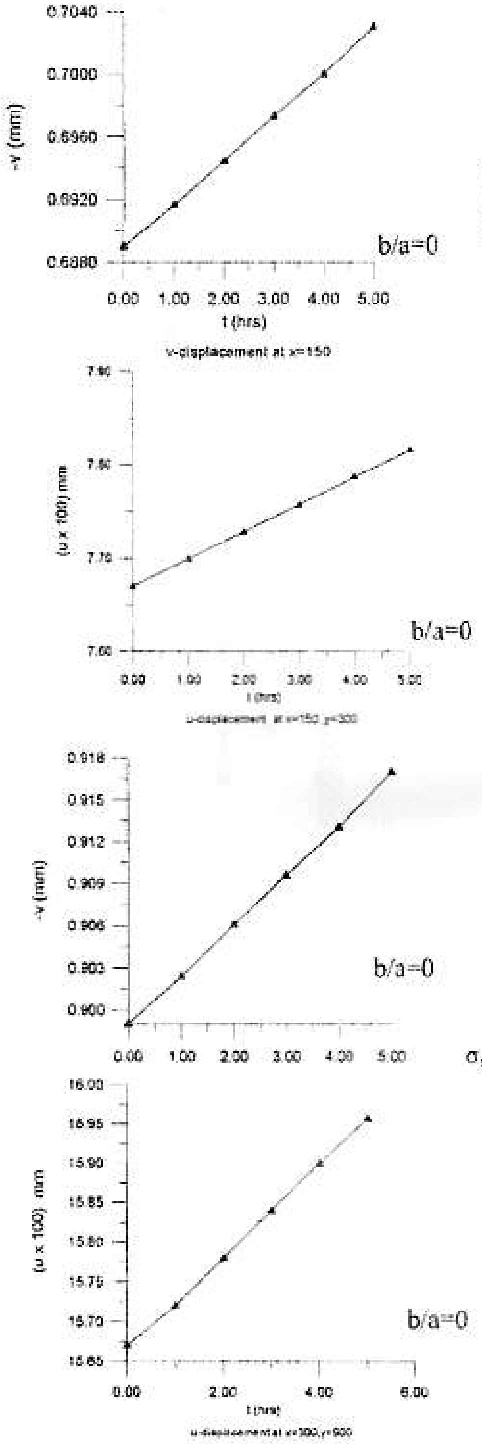


Fig.(4)-(a)

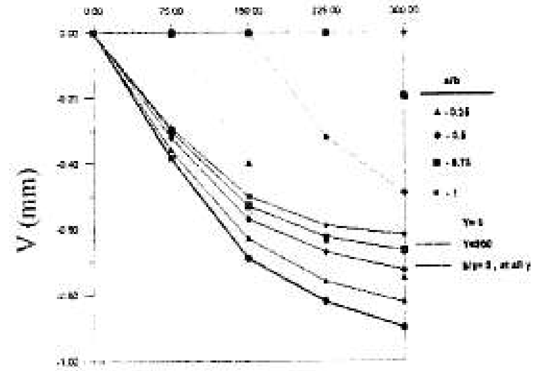
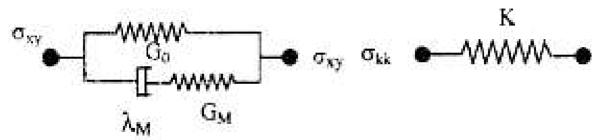


Fig.(4)-(b)

3.3 A simple typical rocket grain

It is a long grain encased in a rigid sheath (plane strain condition). The problem has practical significance in solid propellant analysis where two failure modes occur. first is the separation of the propellant from the case at the cylindrical interface, and the second is the cracking of the propellant material at the inner free face. This problem has been solved by [6] using **ADINA** code ***"A finite element code of a linear thermoviscoelastic materials"***



(a) deviatoric (shear) **(b) volumetric (bulk)**

Fig.(5)

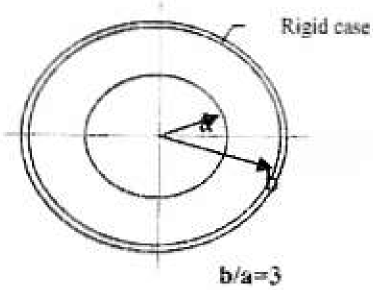
The rheological model of this case is shown in Fig.(5):

The geometrical dimen-tions of the problem and material properties are shown in Fig.(6)

Thermal stresses and strains are reduced to a parametric form by setting the coefficient of thermal expansion $\alpha=1$, and $C_1=C_2=0$
The transient temperature load function is taken as:

$$T = 1 - e^{-t/tm}$$

where tm is the retardation time ($\lambda M/GM$). The variation of the dimensionless tangent stress ($\sigma_\theta/\alpha\ k\ \Delta T$), radial stress ($\sigma_r/\alpha\ k\ \Delta T$), and radial strain ($\varepsilon_r/\alpha\ k\ \Delta T$) with the time are shown in Fig.(7).



$(G_0+G_M)/K=0.0201342$
$Go/G_M=0.1$
Fig.(6) Rocket grain

From the figures the stresses and strains in both codes "FEVES" and "ADINA" are identical (may be due to same formulation and same elements).

Conclusions:

The presented **"FEVES"** code in the paper is based on linear uncoupled thermo-viscoelastic theory. The software is made useful for all permissible values of Poisson's ratio by using a selective integration procedure, a smoothing technique is used in the software (for incompressible or nearly incompressible solids) to get the stresses and strains at the nodal elements. The presented FEM results are very close to the published FEM and analytical results.

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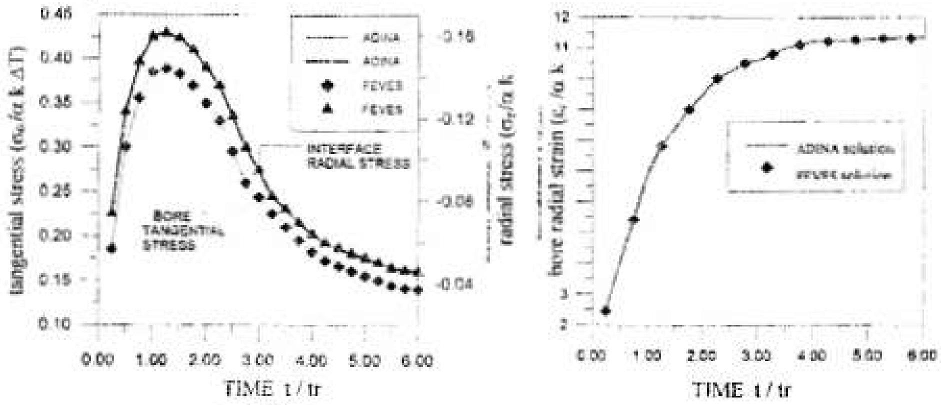

Fig.(7)

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Appendix: (A) Finite Difference Integration

Equation (21) is of convolution integral type, which is required to be integrated by a numerical integration technique such as a trapezoidal rule, the convolution integral term is:

$$L\{\varepsilon(t)\} = \int_{0^+}^{t} \frac{\partial G(\zeta - \zeta')}{\partial t'} \{\varepsilon_{ij}(t')\} dt'$$

Hence

$$L\{\varepsilon(t)\} = \sum_{j=1}^{k-1} \int_{0^+}^{t} \frac{\partial G(\zeta - \zeta')}{\partial t'} \{\varepsilon_{ij}(t')\} dt'$$

Each of the integration is transformed to the finite approximation

$$\int_{t_j}^{t_{j+1}} \{\varepsilon_{ij}(t)\} \frac{G(\zeta_k - \zeta')}{\partial t'} dt'$$

$$= \frac{1}{2} [\varepsilon_{ij}(t_{j+1}) + \varepsilon_{ij}(t_j)] \int_{t_j}^{t_{j+1}} \frac{G(\zeta_k - \zeta')}{\partial t'} dt'$$

$$= \frac{1}{2} [\varepsilon_{ij}(t_{j+1}) + \varepsilon_{ij}(t_j)] [G(\zeta_k - \zeta_{j+1}) - G(\zeta_k - \zeta_j)]$$

From eq(17):

$$= \frac{1}{2} [B] [q(t_{j+1}) + q(t_j)] [G(\zeta_k - \zeta_{j+1}) - G(\zeta_k - \zeta_j)]$$

$$L\{\varepsilon(t)\} = \frac{1}{2} [B] \sum_{j=1}^{k-2} [q(t_{j+1}) + q(t_j)] [G(\zeta_k - \zeta_{j+1}) - G(\zeta_k - \zeta_j)]$$

By applying time step (k-1), the above equation will become:

$$L\{\varepsilon(t)\} = \frac{1}{2} [B] \sum_{j=1}^{k-2} [q(t_{j+1}) + q(t_j)] [G(\zeta_k - \zeta_{j+1}) - G(\zeta_k - \zeta_j)]$$
$$+ \frac{1}{2} [B] [G(0) - G(\zeta_k - \zeta_{k-1})] \{q(t_k)\}$$
$$+ \frac{1}{2} [B] [G(0) - G(\zeta_k - \zeta_{k-1})] \{q(t_{k-1})\}$$

Appendix :(B) Smoothing Technique:

Reference [11] explainas the smoothing technique for 8-node serendipity isoparametric element to convert the stresses at (2×2) Gauss

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{Bmatrix} = \begin{bmatrix} 1+\frac{\sqrt{3}}{2} & -\frac{1}{2} & 1-\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1+\frac{\sqrt{3}}{2} & -\frac{1}{2} & 1-\frac{\sqrt{3}}{2} \\ 1-\frac{\sqrt{3}}{2} & -\frac{1}{2} & 1+\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1-\frac{\sqrt{3}}{2} & -\frac{1}{2} & 1+\frac{\sqrt{3}}{2} \end{bmatrix} \begin{Bmatrix} \sigma_I \\ \sigma_{II} \\ \sigma_{III} \\ \sigma_{IV} \end{Bmatrix}$$

points to the geometrical nodes "***local smoothing***" by using the matrix shown below:

Stresses on the left-hand side represent the four corner nodal stresses, while the stresses on the right-hand side represent the four (2×2) Gauss point stresses.