Large Amplitude Vibrations of Clamped – Free Beams Under Thermal Gradient

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Abstract

The large amplitude vibrations of an elastic straight beam with clamped – free ends which have been subjected to thermal gradient are studied assuming that the beam is undergoing inextensional motion. The rotary inertia and shearing effects are neglected. This is because of their small values. The governing equations are obtained by using Hamilton's principle. The thermal effects on the nonlinear period and frequency ratios are shown through plots. The more effective factor to the nonlinear vibration (large amplitude) is the second moment of area of the beam cross section.

الاهتزاز ذو السعة الكبيرة لعتبة كابولية يؤثر عليها تدرج حراري

الخلاصة

تناول هذا البحث دراسة الاهتزازات الحرة لعتبة كابوليه مرنه تتحرك بسعة عالية يؤثر عليها تدرج حراري بأفتراض ان العتبه لاتحدث فيها أستطالة طولية . ومن أجل تبسيط التحليل فقد أهملت تاثيرات القوى القصية والقوى الدورانية . وقد تم اعتماد مبدأ (هاملتون) في عملية اشتقاق معادلات الحركة. أن تأثير التدرج الحراري على نسبتي الفترة غير الخطية إلى الفترة الخطية والاهتزاز غير الخطي إلى الاهتزاز الخطي قد تم توضيحه خلال الرسوم البيانية . وقد وجد أن العامل الأكثر تأثير الأهتزاز غير الخطي (ذو السعة الكبيرة) هو عزم المساحات الثاني للعتبة.

NOMENCLATURE

A: Area of beam cross section, m^2.

E: Modulus of elasticity, N/m^2.

E_o: Modulus of elasticity at the clamped end, N/m^2.

I: Second moment of area of beam cross section, m^4.

K: Kinetic energy, J.

ℓ: Beam length, m.

t: Time, s.

U: Potential energy, J.

ρ: Density of beam material kg/m^3.

Ψ: Dimensionless temperature of beam.

Ψ_0: Dimensionless temperature of beam at clamped end.

$\Delta\psi$: Dimensionless temperature difference between the ends of the beam.

INTRODUCTION

The theory of vibration of beams is based upon some assumptions such as small dynamic deflections. Recently many authers have presented a simple formulation for non–linear vibrations of beams (such as traction linkages in tractors) by considering the axial and transverse inertias in the equation of

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motion. In 1988, Nageswara and Venkteswaram [1] obtained the formulation for the first mode of vibration of a clamped – free uniform beam. Their results compare well with that presented in 1965 by Wagner [2]. Though the behaviour of the first mode of vibration is of hardening type, the authors noticed that the behaviour of the second mode of vibration is of softening type. An attemp has also been made to study the large amplitude vibrations of free – free uniform beam by following the analysis as described in Ref. [1]. It is found that the first natural frequency of free – free beam decreases with increasing the amplitude of vibration. This phenomenon is observed in experiments performed on large aircraft structures [3,4].

A study on the linear dynamical behavior of non-uniform cantilever beam supported by elastic end support and subjected to axial force and temperature gradient has been achieved [5]. They concluded that increasing the end support stiffness and the temperature gradient shifts the instability frequencies to higher or lower values depending on the corresponding mode. Also they concluded that the temperature gradient has the effect of making the beam more sensitive to periodic loads and hence the beam stability reduces. Azrar, Banamar & White [6] studied the non-linear dynamic response problem of simply – simply supported and clamped – clamped beams when they undergoes large amplitudes of vibration. They used the single mode approach for investigating the geometrical non-linearity on resonant phenomenon.

From the above and other literature which, there is no work deals with the temperature effects on the vibrations of beams when undergo large amplitudes of vibration. In this paper the effect of temperature gradient on the large amplitude vibration of cantilever beams is presented.

ANALYSIS:

The element of the deformed beam is shown in Figure (1). To simplify the anlysis, the following assumptions are imposed:

- No damping is considered, i.e. the system is conservative.
- The temperature gradient along the beam is considerd to be linear.
- The beam is of uniform cross section and mechanical properties.
- According to Hamilton's principle in its simplified form, the kinetic and potential energies, are:

$$K = \frac{\rho A \ell}{2} \int_0^1 (\dot{x}^2 + \dot{y}^2) d\xi \quad (1)$$

$$U = \frac{EI\ell}{2} \int_0^1 k^2(\xi, t) d\xi \quad (2)$$

where, a dot indicates differentiation with respect to time and $\kappa(\xi,t)$ the curvature of the centre line of the beam and it is given by Wagner [2]:

$$\kappa(\xi,t) = \lambda^3 (x'y'' - x''y') \quad (3)$$

where, a prime denotes differentiation with respect to ξ, and $\lambda = 1/\ell$

The Lagrangian function is:

$$L = T - U \quad (4)$$

To apply Hamilton's principle, the following condition should be improved[9]:

$$\delta \int_{t1}^{t2} L dt = 0 \qquad (5)$$

which leads to:

$$\Lambda = \int_{t1}^{t2}\int_{0}^{1} [\rho A(\dot{x}^2 + \dot{y}^2) - EI\lambda^6 (x'y'' - x''y')^2] d\xi dt \qquad (6)$$

in which Λ should be minimum. The variational Λ is written as[2]:

$$\Lambda = \int_{t1}^{t2}\int_{0}^{1} F(\xi, t, x(\xi,t), y(\xi,t), \dot{x}, \dot{y}, x', y', x'', y'') d\xi dt$$

The corresponding Euler equations are:

$$\frac{\partial F}{\partial x} - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{x}}\right) - \frac{\partial}{\partial \xi}\left(\frac{\partial F}{\partial x'}\right) + \frac{\partial^2}{\partial \xi^2}\left(\frac{\partial F}{\partial x''}\right) + \frac{\partial^2}{\partial \xi \partial t}\left(\frac{\partial F}{\partial \dot{x}'}\right) = 0 \qquad (8)$$

and

$$\frac{\partial F}{\partial y} - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{y}}\right) - \frac{\partial}{\partial \xi}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial^2}{\partial \xi^2}\left(\frac{\partial F}{\partial y''}\right) + \frac{\partial^2}{\partial \xi \partial t}\left(\frac{\partial F}{\partial \dot{y}'}\right) = 0 \qquad (9)$$

The relation between x and y is given by:

$$x'^2 + y'^2 = \ell^2 \qquad (10)$$

By executing the operations in Eqs. (8,9) with the use of Eq. (10), and after some manipulations, the following equation could be concluded:

$$\left(\frac{d^2 y}{dt^2} + \gamma^2 \frac{d^4 y}{d\xi^2}\right)\left(1 - \lambda^2 \left(\frac{dy}{d\xi}\right)^2\right)^2 = \lambda^2 \gamma^2 \left[2\left(\frac{dy}{d\xi}\right)\left(\frac{d^2 y}{d\xi^2}\right)\left(\frac{d^3 y}{d\xi^3}\right) + \left(\frac{d^2 y}{d\xi^2}\right)^3 + \lambda^2 \left(\left(\frac{dy}{d\xi}\right)^2 \left(\frac{d^2 y}{d\xi^2}\right)^3 - 2\left(\frac{dy}{d\xi}\right)^3 \left(\frac{d^2 y}{d\xi^2}\right)\left(\frac{d^3 y}{d\xi^3}\right) \right) \right] \qquad (11)$$

The above equation is a non-linear differential equation which cannot be solved simply by the conventional methods.

where $\gamma^2 = \dfrac{EI}{\rho \ell^4 A}$

To account for the temperature effect, the temperature function based on reference temperature, is [5]:

$$\psi = \psi_0 (1 - \xi) \qquad (12)$$

hence Young's modulus is [5]:

$$E(\xi) = E_0 [1 - \beta \psi_0 (1 - \xi)] \qquad (13)$$

Solution of Governing Equation:

The solution of Eq. (11) which is of high nonlinear partial differential equation, can be successful only by means of approximate methods. Shifting the term $(1 - \lambda^2 y'^2)$ to the right side of Eq. (11) and developing it into a power series gives:

$$N[y] = \frac{d^2y}{dt^2} + \sigma^2 \frac{d^4y}{d\xi^2} - \gamma^2 \sum_{\mu=0}^{\infty} \lambda^{2\mu+2} \left[\frac{2dy}{d\xi} \cdot \frac{d^2y}{d\xi^2} \cdot \frac{d^3y}{d\xi^3} + (2\mu+1)(\frac{d^2y}{d\xi^2})^3 \right] (\frac{dy}{d\xi})^{2\mu} = 0$$

(14)

An approximate solution of Eq. (14) can be written as:

$$\bar{y} = \Phi(\xi)f(t) \qquad (15)$$

where, $\Phi(\xi)$ is a given space function and its maximum value is 1, i.e.:

$$|\Phi(\xi)| = 1 \qquad (16)$$

It is known that the exact solution of Eq. (14) cannot be of the product type indicated by Eq. (15).

Thus we proceed to impose the following requirement in the factor $\bar{y}$ on:

$$\int_0^1 \Phi(\xi)N[\bar{y}(\xi,t)d\xi] = 0$$

(17)

This integration yields a nonlinear ordinary differential equation. For a cantilever beam, the boundary conditions are:

$$\Phi(0) = \Phi'(0) = \Phi''(1) = \Phi'''(1)$$

(18)

Now inserting Eq. (15) into Eq. (17) yields:

$$\alpha_0 \ddot{f} + \gamma^2 \sum_{\mu=0}^{\infty} \alpha_{2\mu} + 2 f^{2\mu+1=0} = 0$$

(19)

with:

$$\alpha_0 = \int_0^1 \Phi^2(s)d\xi \qquad (20)$$

$$\alpha_2 = \int_0^1 q\Phi^{iv} d\xi \qquad (21)$$

$$\alpha_{2\mu+2} = -\lambda^{2\mu} \int_0^1 \left[\begin{matrix} 2\Phi\Phi'\Phi''\Phi''' + \\ (2\mu-1)\Phi\Phi'''^3 \end{matrix} \right] \Phi'^{2\mu-2}$$

(22)

This equation can be transformed by means of partial integration as:

$$\alpha_{2\mu+2} = -\lambda^{2\mu} \left\{ \begin{matrix} -[\Phi\Phi'^{2\mu-1}\Phi''^2]_0^1 + \\ \int_0^1 \Phi'^{2\mu}\Phi''^2 d\xi \end{matrix} \right\}$$

(23)

NONLINEAR FREQUENCY:

The equation of motion of a one-mass system with a nonlinear restoring force is:

$$\ddot{f} + F(f) = 0 \qquad (24)$$

where F(f) is a polynomial of f.

Now, according to Atkinson's superposition method of the frequencies, the frequency of the periodic solution of (24), ω, is:

$$\omega^2 = \sum_{\mu=0}^{N} \omega_\mu^2 \qquad (25)$$

where ω_μ is the frequency of a corresponding one – term differential equation:

$$\ddot{f} + \delta_\mu f^{2\mu+1} = 0, \qquad (26)$$
$$(\mu = 0,1,2,\ldots)$$

The natural frequency corresponding to Eq. (26) is given by MacDuff & Curreri [7]

$$\omega_\mu^2 = \pi\delta_\mu(\mu+1)a^{2\mu}\frac{\Gamma^2\left(\frac{\mu+1}{2\mu+2}\right)}{\Gamma^2\left(\frac{1}{2\mu+2}\right)} \qquad (27)$$

where a is the amplitude and Γ is gamma – function which is written as:

$$\Gamma(\sigma+1) = \int_0^\infty x^\sigma e^{-x}dx, (\sigma > 0) \qquad (28)$$

and,

$$\delta_\mu = \frac{\alpha_{2\mu+2}}{\alpha_0}\gamma^2, (\mu \geq 0) \qquad (29)$$

inserting Eq. (27) and Eq. (29) into Eq. (25) yields:

$$\omega^2 = \omega_0^2\pi\sum_{\mu=0}^{N}(1+\mu)I_\mu\beta^{2\mu}\left[\frac{\Gamma\left(\frac{\mu+2}{2\mu+2}\right)}{\Gamma\left(\frac{1}{2\mu+2}\right)}\right]^2\left(\frac{a}{l}\right)^{2\mu} \qquad (30)$$

and the ratio of the nonlinear period T to the linear period T_o can be taken as:

$$T/T_o = f(a/\ell) \qquad (31)$$

where, I_μ and β are numerical values depending on end conditions for clamped – free. The following values are taken from Singirsu [8], α=0.7340955, β=1.8751041.

Discussion

The vibrations of elastic beams with clamped-free ends undergoing large amplitude deflections under the effect of thermal gradients have been proposed. The relation between the vibration amplitude at the beam tip to beam length ratio on the frequency ratio is shown in Fig.(2). This figure shows that at small amplitudes the frequency ratio is about unity and increases with increasing the amplitude of vibration. Throughout the presented results, a hardening spring has effect for the first mode, whereas a softening type of non-linearity is noted for the second mode. It is noticed that the spring hardening or softening due to the axial inertia of the beam, depending on the mode shape. It can be seen also from these figures that the non-linear bending stress exhibits a higher increase near the clamped edge, compared with that expected in linear theory.

The effect of temperature gradient on the ratio of the nonlinear frequency to the linear frequency is shown in Fig.(3). It can be shown that increasing the temperature gradient through the beam length will lead to increases in the frequency ratio. This effect is increased with increasing the amplitude ratio. In other words, the temperature gradient has no effect at zero- amplitudes, but a very clear effect at large amplitudes.

The effect of temperature gradient on the relation between the ratio of nonlinear period to the linear period" versus "the vibration amplitude to beam length ratio" is shown in Fig. (4). It can be concluded that the time ratio decreases with increasing the temperature gradient for the first mode, but increase for the second mode. This phenomenon is due to two reasons; Firstly; the temperature gradient causes a change in the modulus of elasticity proportional to the value of the temperature gradient. The change of the modulus of elasticity along the beam causes an internal force which affects both of the period and frequency ratios. The second reason is that the temperature gradient causes a thermal stresses which have positive or negative effect on the frequency and time ratios depending on the type of the stress.

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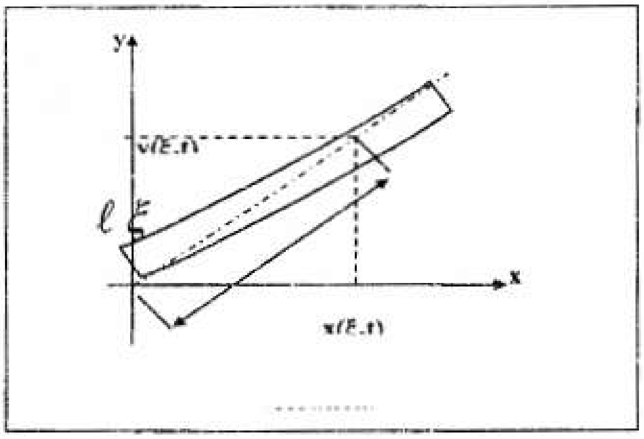
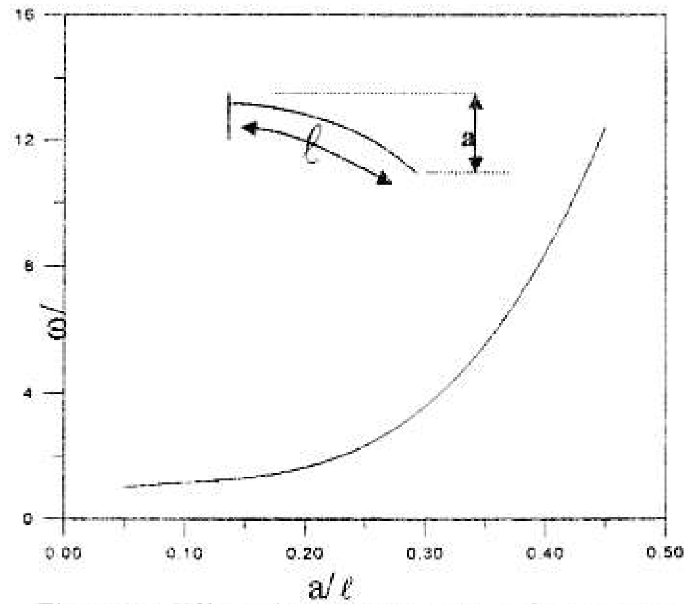
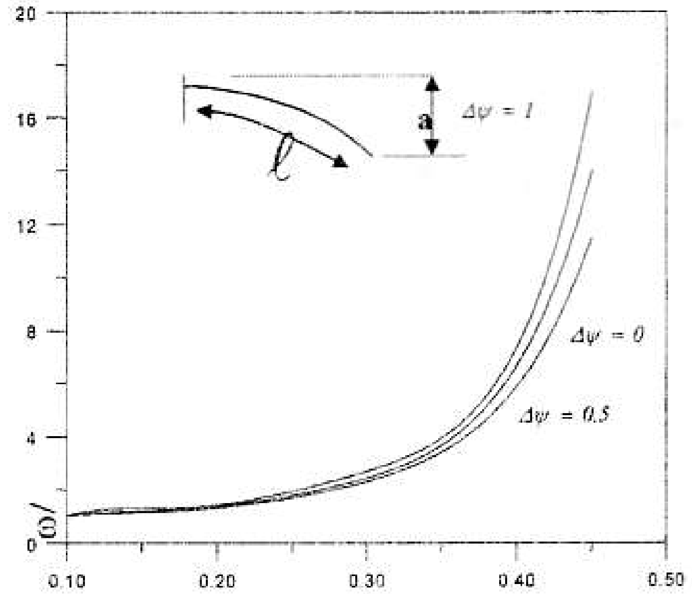


Fig.(1): Representation of the beam during vibration.



Figure(2): Effect of amplitude ratio on frequency ratio, temperature gradient is zero .



Figure(3): Effect of Temperature gradient on frequency ratio.

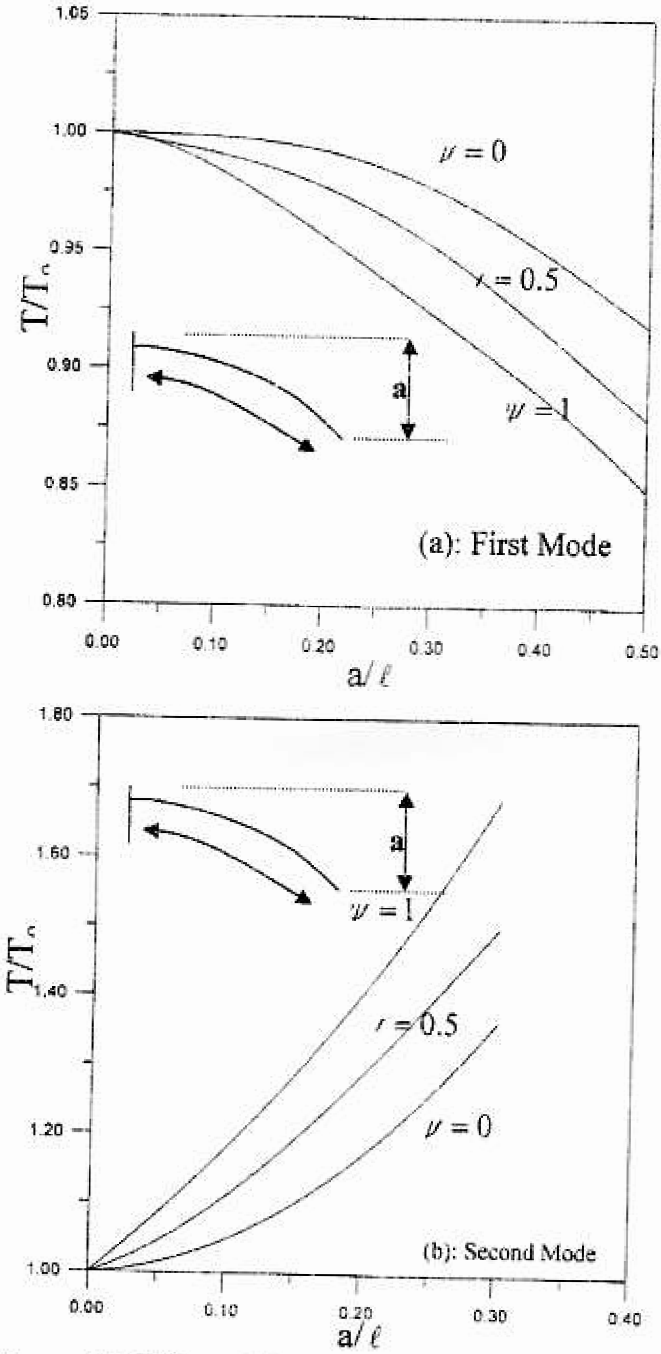


Figure (4): Effect of Temperature gradient on time ratio.