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Analysis of Beams on Non-Linear Elastic Winkler Foundation

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Abstract

This paper deals with thin beams resting on non-linear Winkler foundations by using the secant modulus approach to specify the modulus of subgrade reaction. The finite differences method is used to solve the obtained governing differentional equation. Results of plate loading test of soil obtained in Baghdad are used in the present analysis. The results are compared with those of the elastic Winkler foundations analysis

Keywords:
Beams, finite differences, Winkler foundation, non-linear analysis, secant modulus.

تصرف اعتبات المسندة على الاسس المرنة اللاخطية

الخلاصة

يتناول هذا البحث دراسة تصرف العوارض النحيفة المستندة على الاسس المرنة اللاخطية باستخدام طريقة القاطع لتحديد معاير رد فعل التربة. تم استخدام طريقة العناصر المحددة لحل المعادلات التفاضلية الحاكمة. تم اعتماد نتائج فحص الصفيحة لنموذج من تربة بغداد. تمت مقارنة النتائج مع حالة الأسس الخطية المرنة.

Introduction

Foundation is a structural element, usually soil, to support the overlying structure. It is a load-bearing medium that supports the structure. There are two basic types of elastic Winkler foundation [1]:

- **Type 1:** is characterized by the fact that the resisting pressure in the foundation is proportional at every point to the deflection at that point. It is independent of deflections or pressures produced elsewhere in the foundation (discrete spring system).
- **Type 2:** is defined by the elastic solid continuum which is in contrast to the first type.

In 1946 E. Winkler assumed that the reaction forces of foundation on a supported beam are proportional at every point to the deflection of the beam at that point. This assumption in spite of its simplicity may not represent accurately the actual conditions existing in soil foundation. Beside many classical research work exist [2], a number of recent research works have been developed in this subject such as Hussain [3] Yin [4,5] and Chen [6].

Scope

The object of the present work is to analyze beams on non-linear Winkler's springs making use of the results of plate loading test obtained from field test in Iraq. These results are used to evaluate the modulus of subgrade reaction at different load stages. The obtained values of subgrade reaction are used to analyze two load cases of beams. The results

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are finally compared with the linear elastic analysis of the same beams.

Modulus of Subgrade Reaction

The modulus of subgrade reaction is a conceptual relationship between soil pressure and deflection. It can be determined from plate bearing test that is normally plotted in the form of a pressure-settlement diagram as shown below in figure [1] where:

$$K_S = \frac{q}{\delta} \quad (1)$$

where:

K_S= modulus of subgrade reaction, (force/area/deflection).

q = pressure (force/area).

δ = settlement (length).

For a beam on elastic foundation:

$$K_S\,(\text{beam}) = K_S\,(\text{plate}) \times \text{width of beam} \quad (2)$$

Thus K_S for beam has units of (force / length / deflection)

According to Winkler's hypothesis for linear soil behavior, the slope of load-settlement diagram is constant and has the value of K_S. Since the soil behavior is usually far from being linear. One can approximate K_S value by using one of the followings:

Initial tangent modulus: is the slope of the tangent to the pressure - settlement curve relation at the origin.

Secant Modulus: is the slope of the secant from the origin corresponding to a certain settlement.

However, since the secant value would be more definite at a point than a tangent, in the matter of drawing, K_S is normally taken as the secant modulus corresponding to settlement of 1.25mm for the standard plate loading test [1].

In the present work the value of K_S is specified at different load stages using the secant method. These values are used to solve the differential equation of beams on elastic foundations. An iteration procedure was carried out to get the deflection values at different load levels. When a tolerance of deflection value of 10-5 is obtained, the iteration is stopped and the values of deflection, shear and bending moments are calculated.

The soil used in this work is located in Baghdad [7]. The test was carried out by the College of Engineering of the University of Baghdad / the Consultant Engineering Bureau / as part of series of tests for the soil investigations of the big Baghdad mosque.

Differential Equation

For a beam element supported by a load bearing medium (figure [3]), the following fourth order differential equation is obtained [1]:

$$\frac{d^4y}{dx^4} + \frac{K_s y}{EI} = \frac{q}{EI} \quad (3)$$

where:

y= y(x) is the deflection of the beam.

q= q(x) distributed load (load per unit length).

EI= flexural (bending) rigidity of the beam.

Here, in equation (2) the modulus of subgrade reaction K_S depends on the deflection y.

Method of solution

Although there are many methods used to solve equation (2), the finite differences method [8] is used to solve the governing equation. In this method, the numerical solution of the differential equation depends on obtaining numerical values at some pivotal points spaced in the x-y plane. The derivatives required to obtain the pivot points are approximated either by the derivatives of the nth degree parabola passing through a certain number of pivot points, or by Taylor series expansion as shown in figure [4]. Figure (5) gives the grid used in the finite difference to model the beam.

The expressions used in the current study are:

$$\frac{dy}{dx} = \frac{y_{i+1} - y_{i-1}}{2\Delta x} \text{ (central)} \qquad (4)$$

$$\frac{d^2y}{dx^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2} \qquad (5)$$

$$\frac{d^3y}{dx^3} = \frac{y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}}{2h^3} \text{ (central)} \qquad (6)$$

$$\frac{d^4y}{dx^4} = \frac{y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}}{\Delta x^4} \qquad (7)$$

At (k+1) cycles (or iterations), Ks(k) is estimated from deflections yi(k) of the previous cycle. The general finite difference expression is:

$$\frac{y_{i-2}^{(k+1)} - 4y_{i-1}^{(k+1)} + 6y_i^{(k+1)} - 4y_{i+1}^{(k+1)} + y_{i+2}^{(k+1)}}{\Delta x^4} + \frac{K_{si}^{(k)}}{EI} y_i^{(k+1)} = \frac{q_i}{EI} \qquad (8)$$

Four boundary conditions are required to solve equation (8) as follows:

$y_0 = y_n = 0$

$y_{-1} = y_1$

$y_{n-1} = y_{n+1}$

The solution starts with initial modulus Ksi(0) for node i and the solution is obtained for yi(k). The solution is repeated by using Ksi(1) for node i to obtain yi(2) and so on. At the start, Ksi(0) is the same for all nodes i but then Ksi(1), Ksi(2), Ksi(3),.... for node i will be different for different nodes as yi(k), yi(2), yi(3),.... are different for different nodes.

Applications

Two case studies are considered in this work. First, a simply supported beam subjected to a concentrated load and the second is also a simply supported beam subjected to a uniform load as shown in figure [6].

Results

For the simply supported beam, figure (7) shows the deflection profile along x-direction for the linear elastic and non-linear elastic Winkler foundation while figures (8) and (9) show the bending moment and shearing force along x-direction. The results show nonlinear effect of Ks in the two solutions. For the beam under a concentrated load, figures (10), (11) and (12) show the deflection profile, bending moment diagram and shearing force diagram. The results show nonlinear effect of Ks in the two solutions. Figure (13) and table (1) show that the mid-span deflection for the linear and nonlinear modulus decreases as the depth of the beam increases because the section flexural rigidity EI of the beam increases. Figures (14) and (15) and tables (2) and (3) show that the mid-span moment and maximum shear force increase as the depth of the beam increases because also the section flexural rigidity EI of the beam increases for the two approaches (linear and non linear).

Conclusions

1. The results from the linear and non-linear solutions show rather different values for both deflection and bending moment

but rather close values for shear force for high load applied on the beam rested on elastic foundation.

2. In order to get scale magnification and differences in results, high applied was used in the two case studies. Ordinary loads give no significant differences between linear and non-linear behavior of soil.
3. It is obvious that the elastic method for analyzing beam resting on elastic medium (Winkler's assumption) is still valid for ordinary applied loading on beams. The nonlinear behavior of soil was adapted (adjusted) by using high applied loads (to magnify the scale and the difference in results).
4. The effect of beam depth on maximum beam deflection and bending moment is found to be significant but not much on shearing force.

References

1. Bowels, J.E. "Foundation Analysis and Design", McGraw-Hill 1988.
2. Hetenyi, M. "Beams on Elastic Foundation" The University of Michigan press. Ann Arbor, Mish. (Jan. 1946).
3. Husain, H.M., Al-Jubory, A. "Numerical Solutions of Deep Beams under Generalized Loading", Journal of Engineering and Technology, Baghdad, Vol.15, No.10, 1995.
4. Yin,J. H, "Comparative modeling study of reinforced beam on elastic foundation" Journal of Engineering Mechanics, March 2000.
5. Yin,J. H, "Closed-form solutions for Timoshenko beam on elastic foundation" Journal of Engineering Mechanics, August 2000.
6. Chen, C.N.,"Solution of beam on elastic foundation by DQEM", Journal of Engineering Mechanics, December 1998.
7. Soil Investigation Report of Big Baghdad Mosque.
8. Salvadori, M.G., and Baron M. L., "Numerical Methods in Engineering" Printice-Hall, INC, 2nd edition 1961.

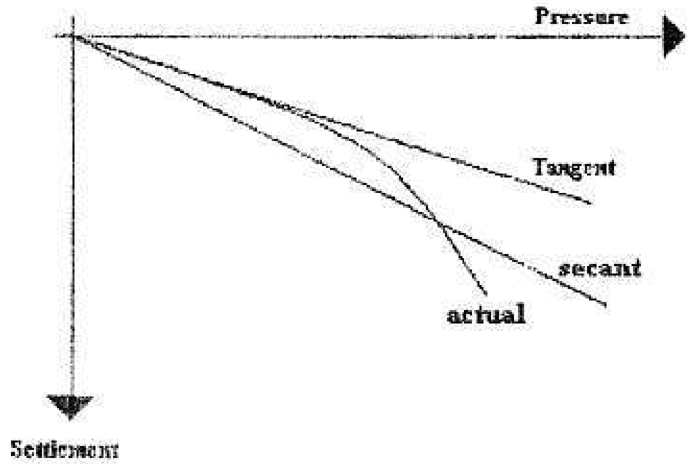

Figure (1) Typical load-bearing curve

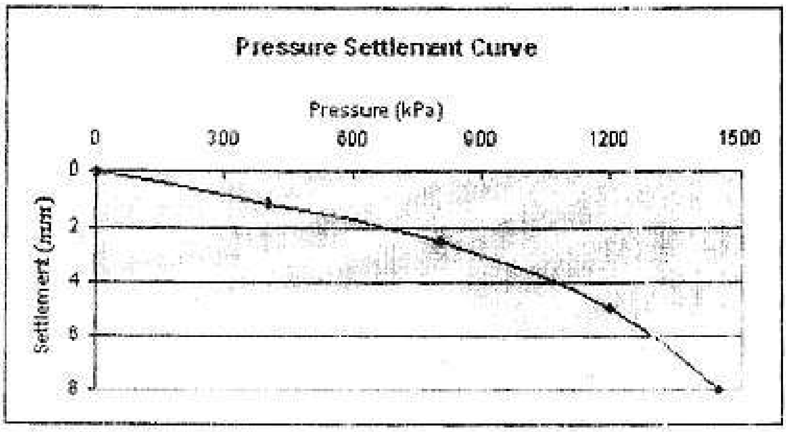

Figure (2) Pressure settlement curve

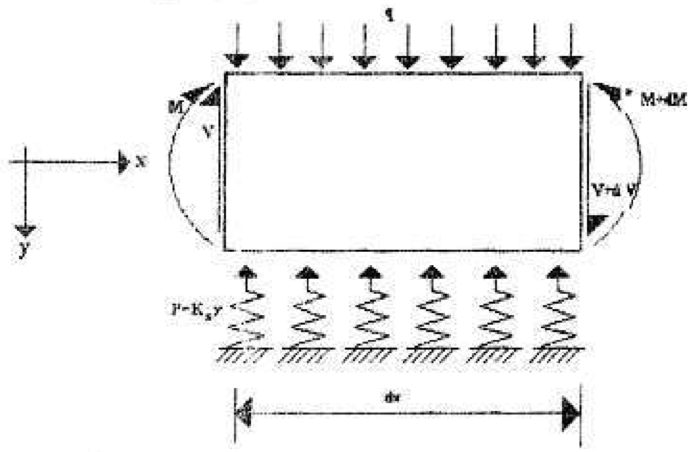

Figure (3) Beam element under applied load

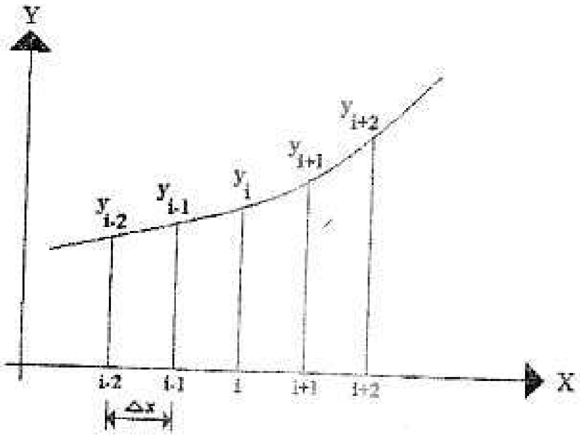

Figure [4] Interpolating parabola

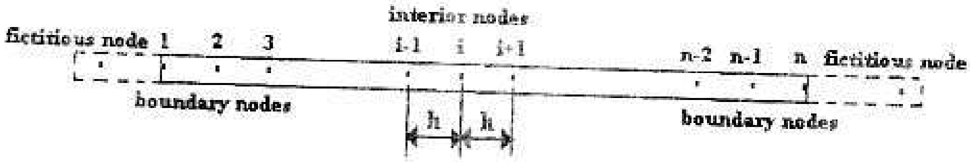

Figure [5] Finite differences mesh

Case 1
Depth h= 0.3 m
Width b= 0.3 m
P= 250 kN
Length= 4 m
E=23500 MN/m2
Ks(0) = 0

Case 2
Depth h= 0.3 m
Width b= 0.3 m
q = 250 kN/m
Length= 4 m
E= 23500 MN/m2
Ks(0) = 0

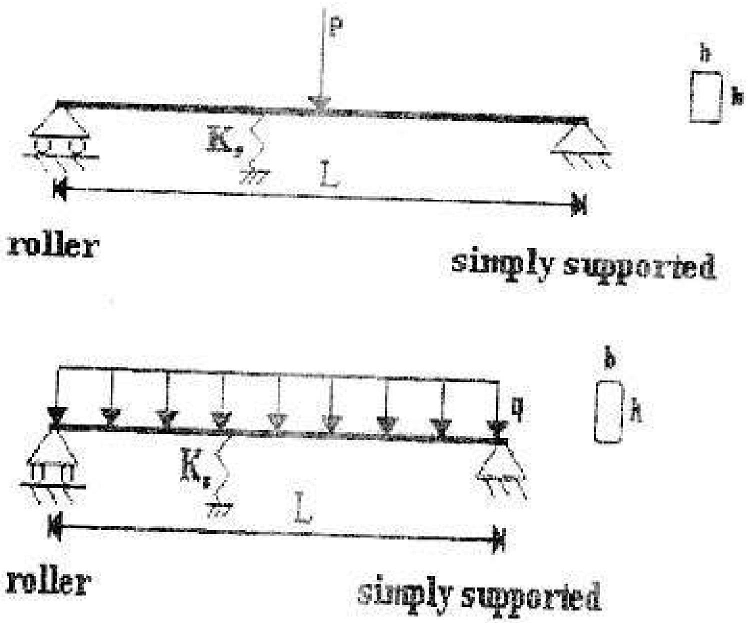

Figure [6] Case studies

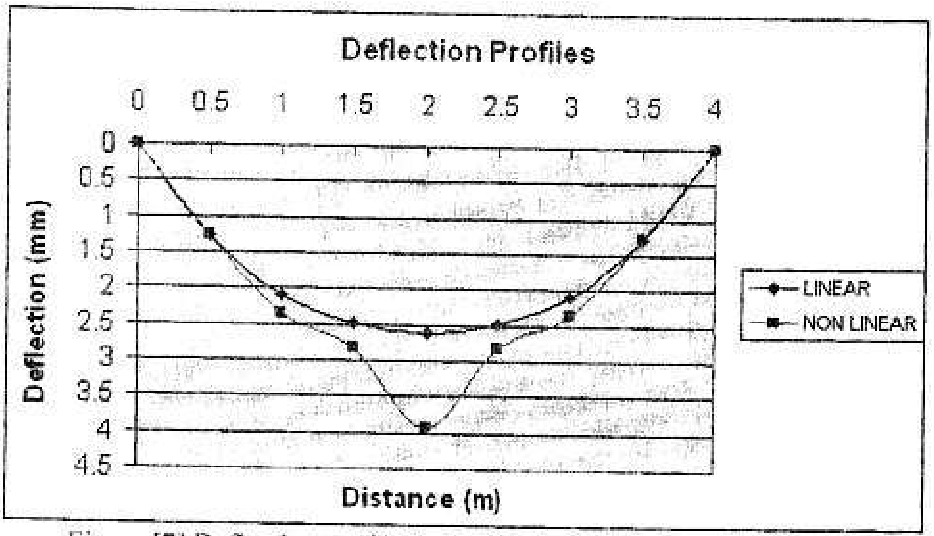

Figure [7] Deflection profile for beam under uniform load (case 1).

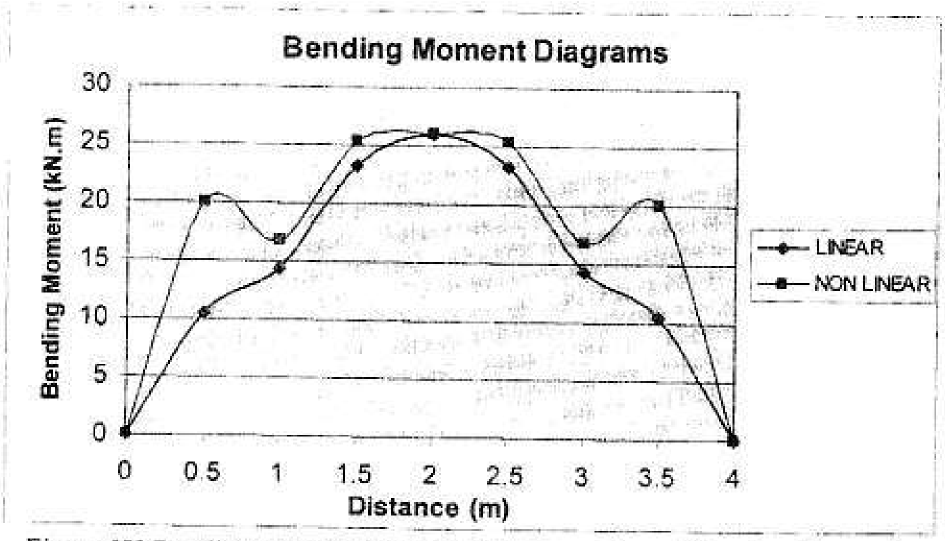

Figure [8] Bending moment diagram for beam under uniform load (case 1).

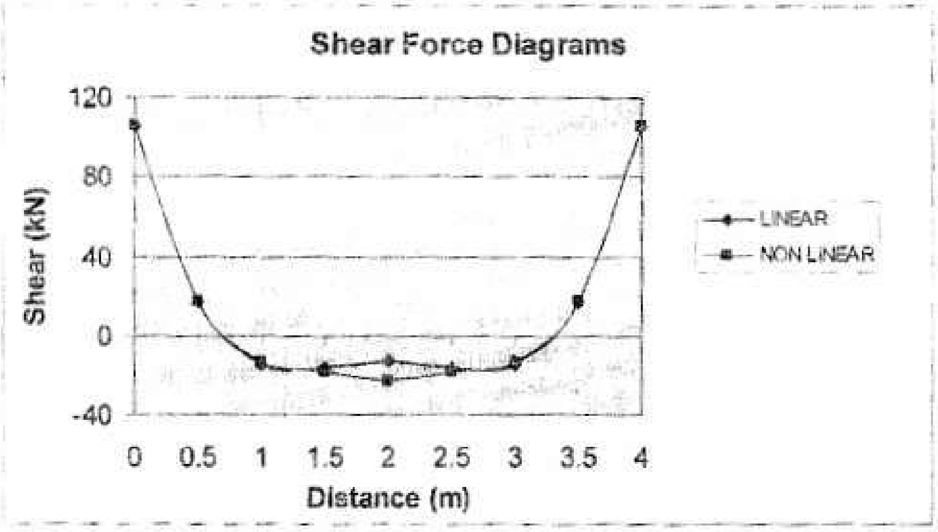

Figure [9] Shearing force diagram for beam under uniform load (case 1).

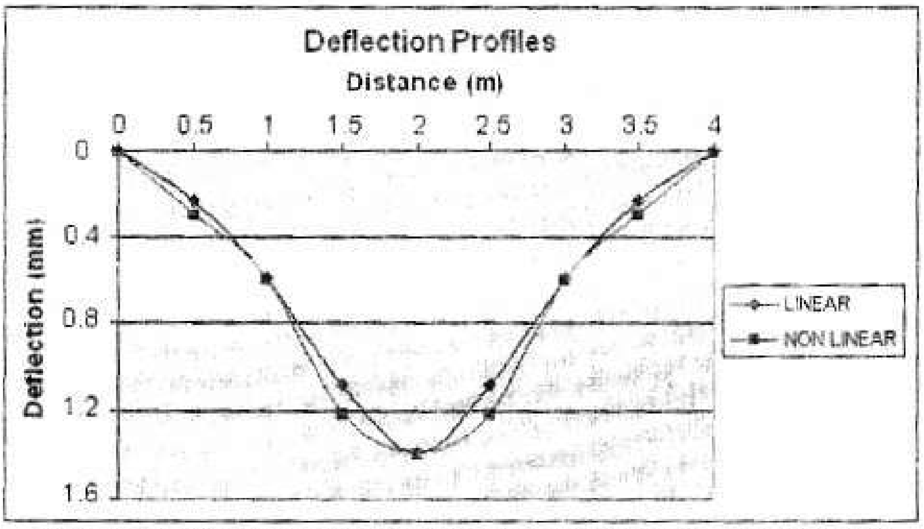

Figure [10] Deflection profile for beam under concentrated load (case 2).

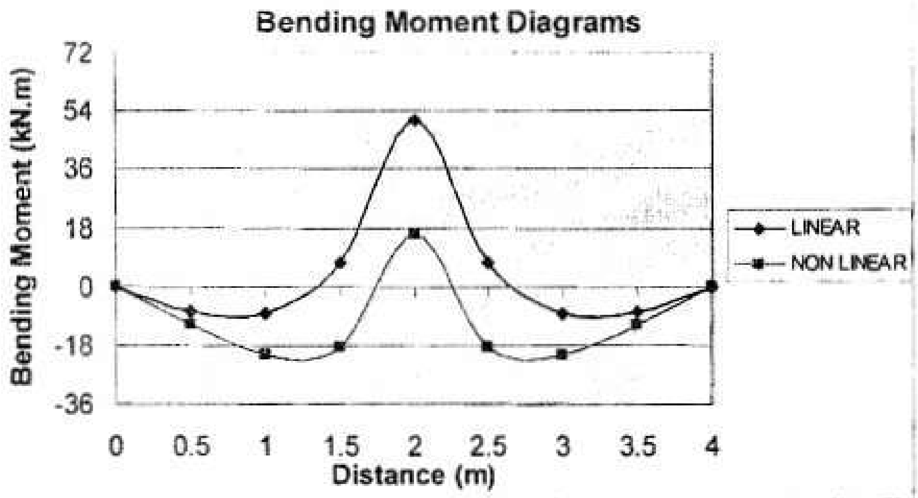

Figure [11] Bending moment diagram for beam under concentrated load (case 2).

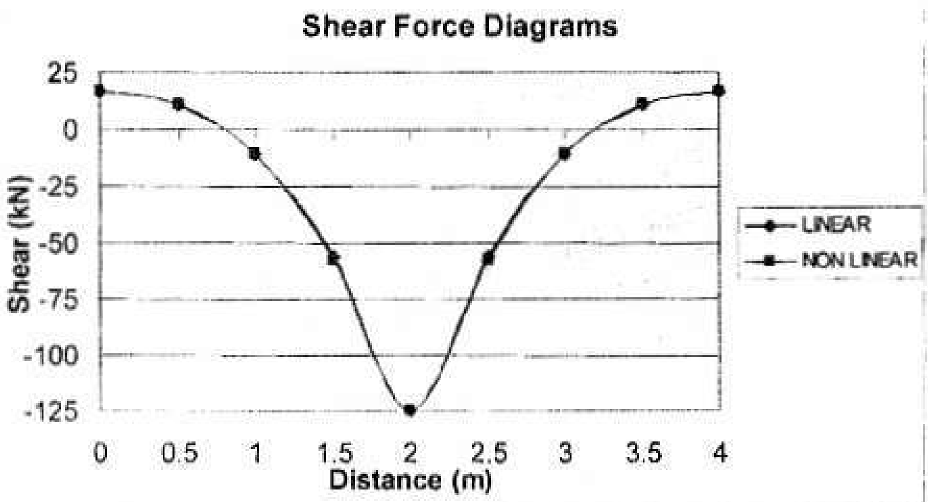

Figure [12] Shearing force diagram for beam under concentrated load (case 2).

Table [1] Effect of beam depth on maximum deflection (mm)

Beam Depth (m)	Linear	Nonlinear	Percentage
0.3	2.6	3.926	51
0.6	2.091	2.293	9.66045
0.9	1.275	1.348	5.72549

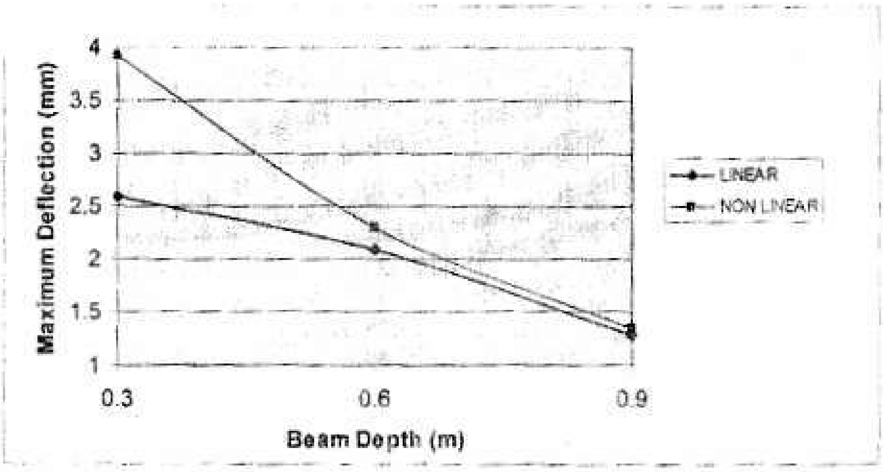

Figure [13] Effect of beam depth on maximum deflection

Table [2] Effect of beam depth on maximum moment (kN.m)

Beam Depth (m)	Linear	Nonlinear	Percentage
0.3	25.947	25.948	0.00385
0.6	113.50	125.113	10.2317
0.9	237.504	250.580	5.50559

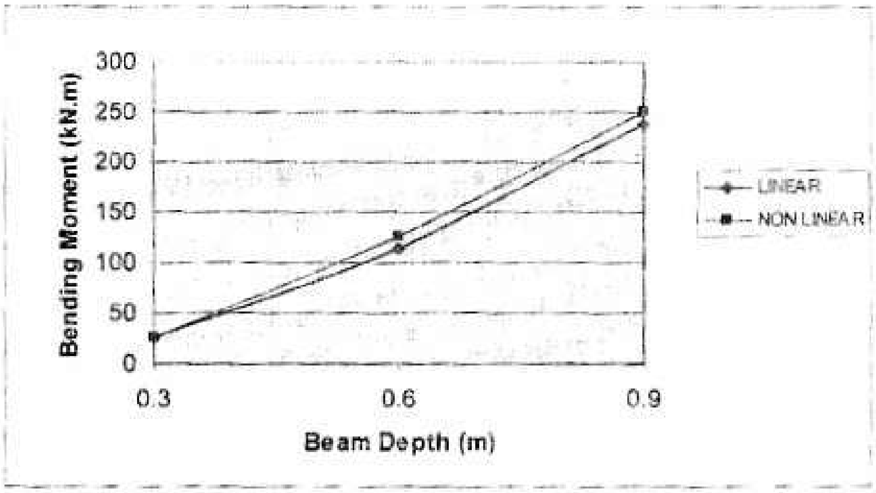

Figure [14] Effect of beam depth on maximum moment

Table [3] Effect of beam depth on maximum shear force (kN)

Beam Depth (m)	Linear	Nonlinear	Percentage
0.3	105.627	105.628	0.00095
0.6	179.186	179.188	0.00112
0.9	177.44	177.45	0.00584

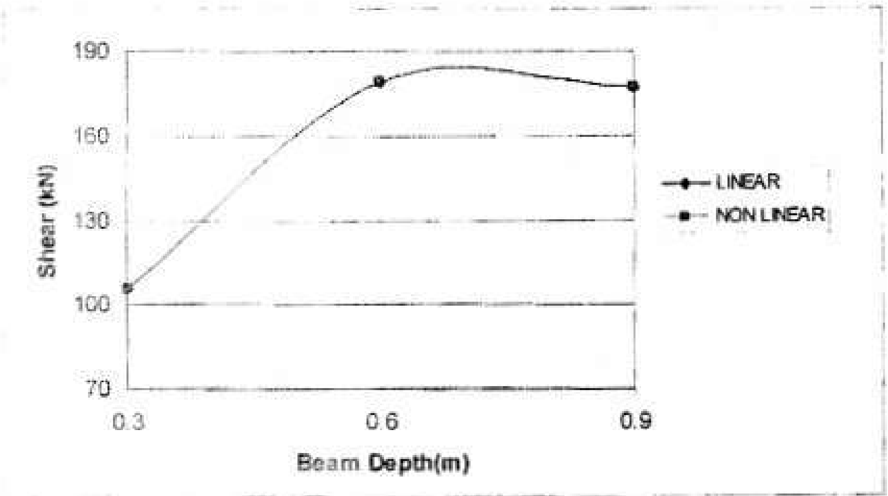

Figure [15] Effect of beam depth on maximum shear force