Eng.& Technology. Vol.24, No.5, 2005

Dynamic Analysis for Soil-Structure Systems Using Dynamic Stiffness

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Received on:21/5/1995
Accepted on:12/7/2005

Abstract

This paper gives exact dynamic stiffness coefficient for beam element with axial force embedded in elastic medium having normal and tangential soil reactions. These moduli of subgrade reactions are assumed to be constants along the length of the element. The axial and flexural stiffnesses are assumed to be uncoupled. The paper concerns on finding the natural frequencies and corresponding modal shapes of plane frames partially embedded in the soil. The variation of the eigenvalues and the eigenvectors by the soil modulus are studied. These eigenvalues and eigenvectors are compared with those when static matrix method is used. Finally, the dynamic response of a multi-story reinforced concrete building frame to blast loading are also considered.

التحليل الديناميكي لمسائل التداخل بين التربة والمنشأ باستخدام الصلادة الديناميكية

الخلاصة

يهتم البحث في دراسة تأثير استخدام المعاملات المضبوطة للصلادة الديناميكية للأضلاع الإنشائية المغمورة في التربة والتي تعاني من ردود فعل عمودية ومماسية على هذه الأضلاع. وقد اعتمدت قيم ثابتة لردود الفعل على طول هذه الأضلاع كما اعتبرت الصلادة المحورية مستقلة عن الصلادة الانحنائية. وقد ركز البحث في إيجاد الترددات الطبيعية وأنماط التشوهات المرافقة لكل منها لمجموعة من الهياكل المستوية والمغمورة جزئياً في التربة. وقد أجريت مقارنة للنتائج المستحصلة مع نتائج طريقة الصلادة الاستاتيكية لقيم مختلفة لردود فعل التربة.

1-INTRODUCTION

The determination of the response by mode superposition method requires the evaluation of natural frequencies and corresponding modal shapes.

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https://doi.org/10.30684/etj.24.5.1
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The evaluation of these modal shapes and frequencies, according to the static coefficients becomes more complicated and unmanageable for structures with many degrees of freedom [1]. However, the analysis becomes relatively simple if for each segment of the structure the properties are expressed in terms of dynamic coefficients.

The dynamic stiffness coefficients relate displacements to forces at the nodal coordinates of element. The difference between the dynamic and static coefficients is that the dynamic coefficients refer to nodal forces and displacements that vary harmonically with time [2]. Moreover, in the dynamic stiffness analysis the stiffness coefficients are functions of the frequency of the motion [3]. The dynamic matrix method enables exact sinusoidally forced or free vibration results to be obtained [4].

In the case of a random excitation, it is necessary first of all to consider free vibration analysis in order to find a set of natural frequencies and their associated modes. The dynamic stiffness matrix [D] is a transcendental function of the natural circular frequency of the structure. Accordingly, the solution of the eigenproblem can be obtained by trial-and-error procedure to find the correct frequencies and the corresponding modal shapes [5].

The present paper uses exact expressions for the dynamic stiffness of a member which is optionally embedded in linearly elastic medium. From these dynamic stiffnesses the overall dynamic stiffness matrix [D] of the frame can be assembled. The normal and tangential soil reactions are taking into account and represented by Winkler model. The paper is concentrated on finding the natural frequencies, modal shapes and the dynamic response of the soil-structure systems under transient loading.

2- THE FINITE ELEMENT MODELING

The part of the frame embedded in the Winkler elastic foundation is represented by linear elastic 2-node beam finite element of uniform cross-section supported by continuous elastic Winkler foundation having normal and tangential moduli of subgrade reactions. Three degrees of freedom are considered at each node; axial displacement, vertical displacement and rotation. The forces and moments at the nodes of each element are expressed in terms of the displacements and rotations at the nodes. Basically this can be achieved in two ways, either by solving the exact dynamic equilibrium equation of motion or by an approximate method such as the use of static equilibrium equation in performing the stiffness matrix. The next two sections give details of these two methods in which beam elements may be considered without damping.

a. Static stiffness method

According to the static equilibrium equations for the beam element, the deflected shape in the flexural case may be written as [6]:

$$\begin{aligned} v(x) = & a_1.\cosh\lambda x.\cos\lambda x \\ & + a_2.\cosh\lambda x.\sin\lambda x \\ & + a_3.\sinh\lambda x.\cos\lambda x \\ & + a_4.\sinh\lambda x.\sin\lambda x \end{aligned} \quad (1)$$

and for axial extension:

$$u(x) = b_1.\cosh \mu x + b_2.\sinh \mu x \quad (2)$$

where a_i & b_i are constants and the parameters $\lambda = (K_n / 4EI)^{0.25}$ and $\mu = (K_t / EA)^{0.5}$.

The constants a_i and b_i may be determined in terms of the end displacements of the element. Using the resulted deflection, the potential and kinetic energies may be written. These energies are then differentiated with respect to each of the nodal displacements to give the static stiffness coefficients of the element. The resulted stiffness equation and the expressions for their stiffness coefficients are given in the appendix.

The continuous mass of the beam finite element is modeled using the improved consistent mass matrix. The expressions for the coefficients of the matrix are given in reference [7]. The stiffness and mass matrices of the structure are assembled from the stiffness and mass matrices of the elements after carrying out the necessary transformation.

b. Dynamic stiffness method

The differential equations from considering the dynamic equilibrium equations of motion of an infinitesimal element are, respectively, for flexural and axial vibrations, given by [5]:

$$\frac{\partial^4 v}{\partial x^4} + \frac{K_n}{EI}.v = -\frac{m}{EI}.\frac{\partial^2 v}{\partial t^2} \quad (3)$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{K_t}{EA}.u = \frac{m}{EA}.\frac{\partial^2 u}{\partial t^2} \quad (4)$$

in which K_n, K_t are, respectively, the normal and tangential soil reactions.

When the separation of variables is used in the solution of each of the above equations, the time independent part of flexural and axial vibrations are given, respectively, as:

$$\frac{d^4V}{dx^4} - \lambda^4 V = 0 \qquad (5)$$

$$\frac{d^2U}{dx^2} + \mu^2 .U = 0 \qquad (6)$$

where

$\lambda^4 = ((m\omega^2 - K_n)/EI)$

and $\mu^2 = ((m\omega^2 - K_t)/EA)$.

The solution of equations (5) and (6) are depend on the sign of parameters λ^4 and μ^2 respectively. For $\lambda^4 > 0$ and $\mu^2 > 0$;

$$V(x) = a_1 \cos \lambda x + a_2 \sin \lambda x + a_3 \cosh \lambda x + a_4 \sinh \lambda x$$

$$U(x) = b_1.\cos \mu x + b_2.\sin \mu x \qquad (8)$$

For $\lambda^4 < 0$ and $\mu^2 < 0$;

$$\begin{aligned} V(x) = {} & a_1.\cosh\bar{\lambda}x.\cos\bar{\lambda}x \\ & + a_2.\cosh\bar{\lambda}x.\sin\bar{\lambda}x \\ & + a_3.\sinh\bar{\lambda}x.\cos\bar{\lambda}x \\ & + a_4.\sinh\bar{\lambda}x.\sin\bar{\lambda}x \end{aligned} \qquad (9)$$

$$U(x) = b_1 \cosh \bar{\mu}x + b_2 \sinh \bar{\mu}x \qquad (10)$$

in which $\bar{\lambda}^4 = -\lambda^4/4$ and $\bar{\mu}^2 = -\mu^2$.

For the case of $\lambda^4 = \mu^2 = 0$, the solutions reduced to their standard forms in static analysis. The dynamic stiffness coefficients for each of the above cases are given in the appendix.

3 DYNAMIC ANALYSIS

a. According to the static stiffness matrix

When the static matrices are used to simulate the structural properties, the dynamic equilibrium equation becomes;

$$[M]\{\ddot{Y}\} + [K]\{Y\} = \{P\} \qquad (11)$$

When the force vector {P} be equal to zero, an important mathematical problem known as an eigenproblem is produced;

$$[M]\{\ddot{Y}\} + [K]\{Y\} = \{0\} \qquad (12)$$

The solution to eq. (12) can be postulated to be of the form;

$$\{Y\} = \{\phi\}\sin\omega(t - t_o) \qquad (13)$$

where $\{\phi\}$ is a vector of the amplitudes of motion. The substitution of eq. (13) into eq. (12) gives;

$$-\omega^2[M]\{\phi\}\sin\omega(t-t_o) + [K]\{\phi\}\sin\omega(t-t_o) = \{0\}$$

Since the sine term is arbitrary and may be omitted, the above equation reduced to the form;

$$\left|[K]-\omega^2[M]\right|\{\phi\}=\{0\} \quad (14)$$

According to the Cramer's rule, the solution of this set of simultaneous equations is of the form;

$$\{\phi\}=\frac{\{0\}}{\left|[K]-\omega^2[M]\right|} \quad (15)$$

Hence a nontrivial solution is possible only when the denominator determinant vanishes. In other words, finite-amplitude free vibrations are possible only when;

$$\left|[K]-\omega^2[M]\right|=0 \quad (16)$$

In general, eq. (16) results in a polynomial equation of degree n in ω^2 which should be satisfied for n values of ω^2. This polynomial is known as the characteristic equation of the system [8]. For each of these values of ω^2 satisfying the characteristic equation, the normal modes which represent the amplitude of deformations of that mode are produced. Accordingly, the complete solution of the eigenproblem requires the evaluation of eigenvalues and the corresponding eigenvectors. In this work, subroutine Jacobi is used for this purpose [1]:

b. According to the dynamic stiffness matrix

The dynamic stiffness coefficient S_{ij} is defined as the harmonic force of frequency ω at nodal coordinate i due to a harmonic displacement of a unit amplitude and of the same frequency at nodal coordinate j. Accordingly, the end forces and moments are related to the end displacements by a single matrix named the dynamic stiffness matrix;

$$[D]\{Y\}=\{P\} \quad (17)$$

where $\{Y\}$ and $\{P\}$ are, respectively, vectors of the amplitudes of nodal displacements and modal forces. In the absence of the externally applied dynamic forces, the structure vibrate freely. Thus, the equation of free vibration of the structure can be written as follows;

$$[D]\{Y\}=\{0\} \quad (18)$$

Similarly, the nontrivial solution of eq. (18) is possible only when;

$$\left|[D]\right|=p(\omega^2)=0 \quad (19)$$

in which $p(\omega^2)$ is the characteristic equation of the system. The above eigenproblem can be solved only by iterative method, because the stiffness coefficients are functions of the unknown frequency. There are many numerical methods used in solving eq. (19). One commonly used and simple technique is the secant method, in which a linear interpolation is employed;

$$\omega_{i+1}^2 = \omega_i^2 - \frac{p(\omega_i^2)}{p(\omega_i^2) - p(\omega_{i-1}^2)}(\omega_i^2 - \omega_{i-1}^2) \quad (20)$$

here ω_i^2 is the eigenvalue for ith iterate. The eigenvalues from static stiffness method are used as a first approximation in eq. (20) to get a second, the second to get the third, and so on. Accordingly the eigenvalues and the stiffness coefficients are adjusted iteratively during the solution. In the present paper, a computer program was developed in solving the eigenproblem.

The forced random vibration of a structure is determined by modal analysis [7&9]. The response analysis by mode superposition requires, first, the solution of the eigenvalues and eigenvectors then the solution of the decoupled equilibrium equations and finally, the superposition of the response in each eigenvector [8].

4 NUMERICAL EXAMPLES

The dynamic analysis of plane frames partially embedded in elastic foundations are investigated. Both the free and the forced vibrations are carried out for the selected case studies. The eigenvalues and the eigenvectors computed by the dynamic matrix method are compared with those found when the static matrix method is used.

Example 1

The dynamic behavior of pile group supporting a vibrating structure is considered. Plane section of the system is shown in Fig.(1) which consists of three piles, embedded in elastic media, with a cap. The normal and tangential soil reactions are taken into account and neglecting the end bearing resistance. System properties are collected in two ways. In the first the static stiffness and mass matrices are used while in the second the dynamic stiffness matrix is considered.

The main variables considered in this problem are the soil reactions. The variation of the natural circular frequencies corresponds to the first four modes are plotted versus the variation of the soil modulus K_n as shown in

Fig.(3). The tangential soil reaction K_t is taken as one-half the normal. The simi-log plot shows that, for low values of K_n, the curves for all values of ω cluster together, and begin to diverge beyond a certain value of K_n. The variation of the first four natural circular frequencies, computed by static and dynamic stiffness methods, with the variation of soil modulus are given in table (1). It is shown that the difference in values of ω between the two methods are increased for higher modes. Moreover, the differences are also increased when the soil modulus is increased.

Table (2) gives the normalized modal shapes corresponding to the first three modes with two values of soil modulus. The values computed according to the dynamic matrix method represent the exact values. The table indicate a quite difference between the two methods.

Example 2

Plane vibration of two-story two-bay reinforced concrete building frame will be considered. The frame is supported on piles embedded in elastic soil. Both normal and tangential soil reactions are taken into account. These reactions are assumed to be constants along the depth of soil. Geometric properties of the frame are shown in Fig.(2-a). It is assumed that the frame has been subjected to the blast loading as shown in Fig.(2-b). The dynamic analysis using dynamic stiffness is carried out to find the eigenvalues, eigenvectors and the structural response to the blast loading. The variations of the eigenvalues for different values of soil modulus are given in table (3). The eigenvectors for the first three modes are given in table (4) for the two methods of analysis. The variations in magnitude of the first four natural frequencies by the variation of soil modulus are given in Fig. (4).

The horizontal displacements with time in nodes 1 and 7 are plotted in Figs. (5 and 6) respectively. Three values of soil modulus are considered as shown in the figures. It is shown that the response of the nodes above the ground are much less affected by the variation of soil reactions than the deflections in the node embedded in the soil.

5 CONCLUSION

This paper gives the exact dynamic stiffnesses for a beam element with axial force embedded in an elastic medium. These stiffnesses enable exact, sinusoidally forced or free vibration results to be obtained.

The validity of the equations presented has been confirmed by comparing the eigenvalues and eigenvectors with those obtained from the static stiffness method. It is found that the errors involved in natural frequencies, when the static matrices are used, are small for different values of soil reactions. However, there is a quite difference in the modal shapes for the two methods of analysis.

The effect of the soil modulus on natural frequencies is also studied. It is concluded that the values of natural frequencies cluster together for low values of soil modulus. Finally, it is shown that the response of the parts above the ground are much less affected by the variation of soil reactions than that in the parts embedded in the soil.

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NOTATIONS

A	Cross sectional area.
a_i, b_i	Constants.
B, D	Constants.
E	Modulus of elasticity.
F_i	Vertical force at node i.
H_i	Horizontal force at node i.
I	Moment of inertia.
K_n	Normal soil reaction.
K_t	Tangential soil reaction.
L	Length of the element.
$p\ (\omega^2)$	Characteristic equation of the system
m	Mass per unit length.
M_i	Moment at node i
n	Number of degrees of freedom.
S_{ij}	Dynamic stiffness coefficient.
u (x)	Displacement functions in axial case.
u_i	Horizontal displacement at node i
U	Time independent part of displ. Function in flexural case.
v(x)	Displacement function in flexural case.
v_i	Vertical displacement at node i
V	Time independent part of displacement function in axial case.
t	Time.
$[D]$	Dynamic stiffness matrix.
$[K]$	Static stiffness matrix.
$[M]$	Mass matrix.
$\{P\}$	Force vector.
$\{Y\}$	Displacement vector.
$\{\phi\}$	Vector of the amplitudes of motion.
θ_i	Rotation at node i.
ψ, ν	Parameters given in the appendix.

ϕ_{ij} Modal shapes.

$\lambda, \bar{\lambda}, \mu, \bar{\mu}$ Parameters.

ω Natural frequency

Table (1) Variation of the first four natural circular frequencies by soil modulus (Example 1).

Soil modulus (Normal) (N/m^2)	Natural frequencies							
	ω_1	ω_1^*	ω_2	ω_2^*	ω_3	ω_3^*	ω_4	ω_4^*
1E2	0.16	0.17	0.21	0.21	0.62	0.59	8.76	8.43
1E3	0.57	0.56	0.66	0.66	1.80	1.85	8.55	8.75
1E4	1.97	1.79	2.09	2.09	5.54	5.51	10.35	11.48
1E5	5.45	5.63	6.53	6.53	12.57	12.56	20.10	26.18
1E6	15.47	17.04	20.20	20.23	29.83	31.15	56.27	78.84
1E7	47.29	49.13	57.39	58.53	81.03	84.73	135.63	140.67
1E8	88.78	133.29	154.83	150.29	189.31	194.30	291.42	295.86
1E9	178.82	299.68	211.21	339.69	437.49	432.26	571.86	564.87
1E10	197.29	773.04	623.59	899.45	695.78	1075.45	1104.77	1305.30

ω_i :Natural frequencies according to the dynamic stiffness method.

ω_i^* : Natural frequencies according to the static stiffness method.

Table (2) Normalized modal shapes corresponding to the first three modes with different values of soil modulus (Example 1).

a- Soil modulus; K_n = 1E2., $K_t = 0.5\ K_n$.

Node	D.O.F.	Normalized modal shapes					
		$\emptyset_{i1}$	$\emptyset_{i1}^*$	$\emptyset_{i2}$	$\emptyset_{i2}^*$	$\emptyset_{i3}$	$\emptyset_{i3}^*$
1	1	0.00484	0.00529	-0.00166	-0.00033	-0.00023	0.00120
	2	0.00240	0.00297	0.00488	0.00518	-0.00631	-0.00578
	3	-0.00011	-0.00066	0.00004	0.00003	0.00158	0.00145
2	4	0.00419	-0.00130	-0.00145	0.00001	0.01557	0.01564
	5	0.00253	0.00428	0.00483	0.00511	-0.00947	-0.00867
	6	-0.00001	-0.00066	0.00001	0.00003	0.00158	0.00144

b- Soil modulus; K_n = 1E6, $K_t = 0.5\ K_n$

Node	D.O.F.	Normalized modal shapes					
		$\emptyset_{i1}$	$\emptyset^{*}_{i1}$	$\emptyset_{i2}$	$\emptyset^{*}_{i2}$	$\emptyset_{i3}$	$\emptyset^{*}_{i3}$
1	1	-0.00554	-0.00548	0.00001	0.0	0.00064	0.00115
	2	-0.00090	-0.00176	-0.00452	-0.00495	-0.00812	-0.00824
	3	0.00026	0.00047	-0.00024	-0.00027	0.00184	0.00182
2	4	-0.00014	-0.00030	-0.00089	-0.00095	-0.00179	-0.00229
	5	-0.00196	-0.00276	-0.00429	-0.00471	-0.00757	-0.00748
	6	-0.00001	-0.00662	0.0	0.00007	0.0	0.00077

$\emptyset_{ij}$; Normalized modal shapes according to the dynamic stiffness method.
$\emptyset^{*}_{ij}$; Normalized modal shapes according to the static stiffness method.

Table (3) Variation of the first four natural circular frequencies by soil modulus (Example 2).

Soil modulus Normal (N/m^2)	Natural frequencies							
	ω_1	ω_1^{*}	ω_2	ω_2^{*}	ω_3	ω_3^{*}	ω_4	ω_4^{*}
1E3	0.57	0.54	0.89	0.87	1.10	1.98	6.31	6.56
1E4	2.12	1.65	2.81	2.80	5.66	5.77	8.78	10.00
1E5	4.68	4.79	8.86	8.85	12.75	12.78	19.68	25.88
1E6	9.85	9.92	27.53	27.56	31.71	31.79	55.59	55.71
1E7	14.04	14.54	53.31	53.56	78.16	79.42	97.46	99.10
1E8	16.93	19.30	56.15	58.10	158.39	163.07	174.83	176.94
1E9	18.84	21.47	57.84	60.63	163.35	165.19	188.89	191.01
1E10	20.05	21.80	59.05	61.06	165.79	165.50	197.10	193.50

ω_i : Natural frequencies according to the dynamic stiffness method.

ω_i^{*} : Natural frequencies according to the static stiffness method.

Table (4) Normalized modal shapes corresponding to the first three modes with soil modulus K_n = 1E4, K_t = 0.5 K_n (Example 2).

Node	D.O.F.	Normalized modal shapes					
		$Ø_{i1}$	$Ø^*_{i1}$	$Ø_{i2}$	$Ø^*_{i2}$	$Ø_{i3}$	$Ø^*_{i3}$
1	1	-0.01034	-0.01036	0.00735	0.00629	0.00148	0.00049
	2	-0.00445	-0.00441	-0.00138	-0.00093	0.00864	0.00782
	3	0.00074	0.00074	-0.00066	-0.00073	-0.00144	-0.00130
2	10	-0.00734	-0.00739	0.00468	0.00372	-0.00425	-0.00468
	11	-0.00445	-0.00441	-0.00138	-0.00143	0.00864	0.00782
	12	0.00075	0.00074	-0.00066	-0.00058	-0.00140	-0.00126
7	19	-0.00439	-0.00426	0.00234	0.00249	-0.00794	-0.00803
	20	-0.00445	-0.00441	-0.00138	-0.00125	0.00864	0.00782
	21	0.00069	0.00081	-0.00051	-0.00044	-0.00052	-0.00048
10	28	0.00005	0.00203	-0.00057	-0.00062	-0.00855	-0.00968
	29	-0.00445	-0.00441	-0.00138	-0.00121	0.00864	0.00782
	30	0.00037	0.00077	-0.00022	-0.00014	0.00019	-0.00011

$Ø_{ij}$; Normalized modal shapes according to the dynamic stiffness method.
$Ø_{i*j}$; Normalized modal shapes according to the static stiffness method.

APPENDIX

The member equation for the beam element with axial force can be written as:

$$\begin{bmatrix} H_1 \\ F_1.L \\ M_1 \\ H_2 \\ F_2.L \\ M_2 \end{bmatrix} = \frac{E}{L}\begin{bmatrix} \xi & o & o & -\eta & o & o \\ o & \gamma & \nu & o & -\varepsilon & \delta \\ o & \nu & \alpha & o & -\delta & \beta \\ -\eta & o & o & \xi & o & o \\ o & -\varepsilon & -\delta & o & \gamma & -\nu \\ o & \delta & \beta & o & -\nu & \alpha \end{bmatrix}\begin{bmatrix} u_1.A \\ v_1.I/L \\ \theta_1.I \\ u_2.A \\ v_2.I/L \\ \theta_2.I \end{bmatrix} \tag{21}$$

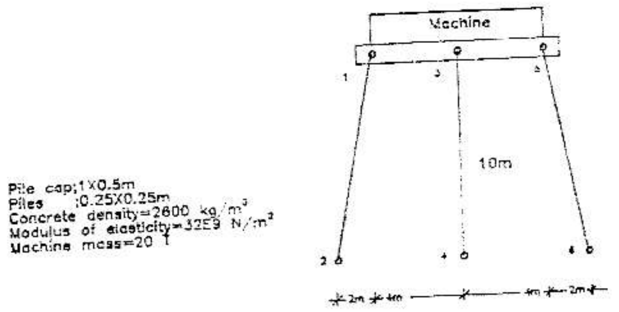
The coefficients $\xi.\eta.\gamma.\nu.\varepsilon.\delta.\alpha.$ and β are given in the tables (5&6).

Table (5) Stiffness coefficients for beam element with; $S = \sin\psi$, $C = \cosh\psi$, $S^* = \sinh\psi$ and $C^* = \cosh\psi$

	Static stiffness coefficients	Dynamic stiffness coefficients ($m\omega^2 - K_n$) >0	<0	=0
γ	$4\psi^2 D(SC + S^*C^*)$	$\psi^2 D(SC^* + S^*C)$	$4\psi^2 D(SC + S^*C^*)$	12
ν	$2\psi D(S^2 + S^{*2})$	$4DSS^*$	$2\psi D(S^2 + S^{*2})$	6
α	$2D(S^*C^* - SC)$	$D(SC^* - S^*C)$	$2D(S^*C^* - SC)$	4
ε	$4\psi^2 D(SC^* + S^*C)$	$\psi^2 D(S + S^*)$	$4\psi^2 D(SC^* + S^*C)$	12
δ	$4\psi DSS^*$	$\psi D(C^* - C)$	$4\psi DSS^*$	6
β	$2D(SC^*-S^*C)$	$D(S^*-S)$	$2D(SC^*-S^*C)$	2
D	$\psi/(S^{*2} - S^2)$	$\psi/(1 - CC^*)$	$\psi/(S^{*2} - S^2)$	-
ψ	$L.(K_n/4EI)^{0.25}$	$L.((m\omega^2-K_n)/EI)^{0.25}$	$L.((K_n-m\omega^2)/4EI)^{0.25}$	-

Table (6) Stiffness coefficients for axial element with; $S = \sin\nu$, $C = \cos\nu$, $S^* = \sinh\nu$ and $C^* = \cosh\nu$

	Static stiffness coefficients	Dynamic stiffness coefficients ($m\omega^2 - K_t$) >0	<0	=0
ξ	C^*B	CB	C^*B	1
η	B	B	B	1
B	ν / S^*	ν / S	ν / S^*	-
ν	$L.(K_t/EA)^{0.5}$	$L.((\omega^2 m-K_t)/EA)^{0.5}$	$L.((K_t-m\omega^2)/EA)^{0.5}$	-



Fig(1) Details of example (1)

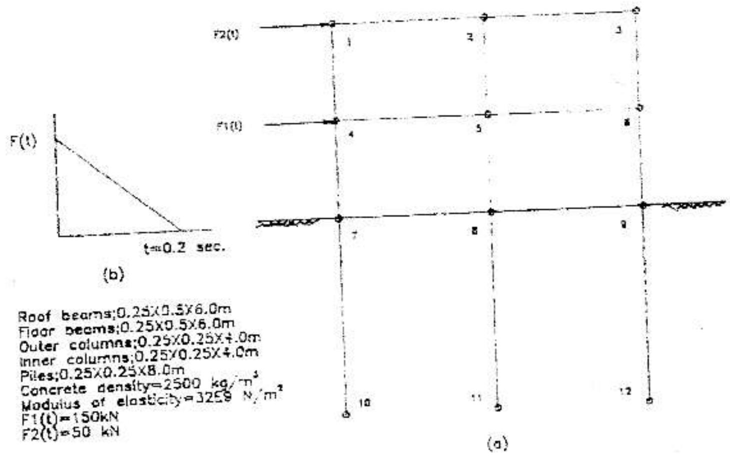


Fig.(2) Details of example (2)
a–structural frame
b–impulsive loading

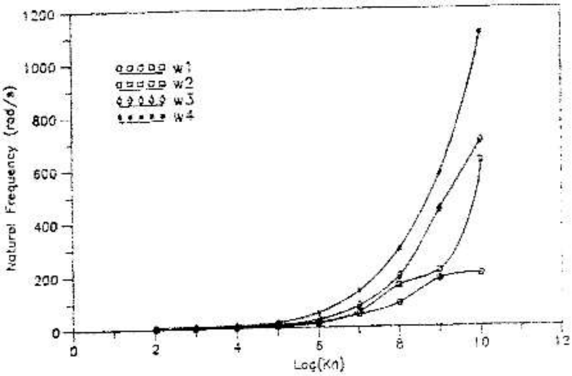


Fig.(3) Variation of the natural circular frequencies by soil modulus.

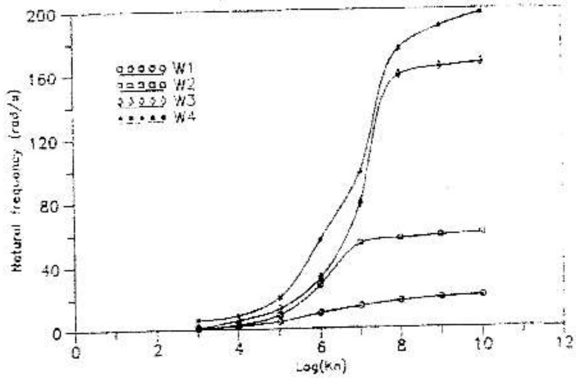


Fig.(4) Variation of the natural circular frequencies by soil modulus.

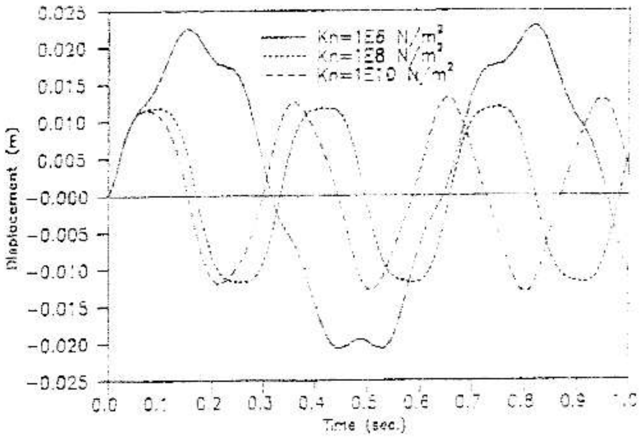


Fig.(5) Horizontal displacement for node 1.

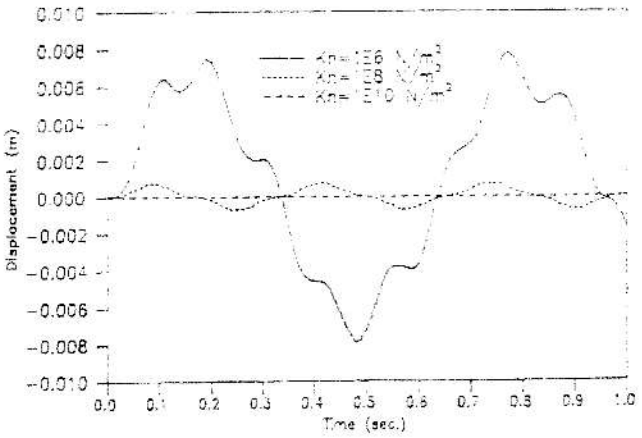


Fig.(6) Horizontal displacement of node 7.