Theorems on Certain Fractional Function and Derivative

Dr. Ahmed Zain al-Abdin
Science College, University of Mosul/ Nanowa
Maha Abd Al-Wahab
Applied Science Department, University of Technology/Baghdad
Email: MAHA101511@yahoo.com

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ABSTRACT

The aim of this paper is to prove the correctness of the relation

\[ I_a^\alpha I_a^\alpha h_\alpha(x) = h_\alpha(x) \quad \text{for all } x \in (a, b], \quad 0 < \alpha \leq 1, \] after we prove the continuity of the fractional function

\[ h_\alpha(x) = \frac{\mu(x-a)^{1-\alpha}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int (x-t)^{\alpha-1} f(g(t))dt \quad \text{in } (a, \infty), \] where \(|h_\alpha(x)| \leq M\)

for all \(x \in (a, \infty)\) and \((M \in \mathbb{R}^+, M > 0)\).

Keywords: Fractional function, derivative.

INTRODUCTION

Fractional Calculus is three Centuries old as the conventional calculus, but not very popular among science and engineering community. This subject was with mathematicians only; in last few years this was pulled to several applied fields of science, economies and engineering. The advantage of fractional derivative apparent in mechanics and stability analysis of fractional control of robotic time delays systems by [1], [2] and in nuclear energy science by [3], [4].
also in the other physical and chemistry fields by [5], [6] and in computer hard disc by control [7]. In this paper we shall prove that:

\[ \begin{align*}
\frac{d^\alpha f}{dx^\alpha} = f^\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt \\
0 < \alpha \leq 1, \text{ after we prove the continuity of the function}
\end{align*} \]

on \((a, \infty)\) for all \(x > a\), \(0 < \alpha \leq 1\), where \(\mu\) is constant, \(f, g\) are continuous functions on \((a, \infty)\).

**THEOREM OF CONTINUITY**

Let \(0 < \alpha \leq 1\) and \(f, g\) be continuous functions on \((a, \infty)\), where \(a \in \mathbb{R}\) and such that

\[ \sup \{ |f(g(x))| : x \in (a, \infty) \} = M < \infty. \]

Define

\[ h_\alpha(x) = f(x) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt \]

\(h_\alpha(x)\) is continuous on \((a, \infty)\).

**Proof**

Clearly \(g_1(x)\) is continuous for all \(x > a\).

Let

\[ y(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt \]

for all \(x \geq a\) \(\ldots (2.1)\)

and if \(x_1, x_2 \in [a, \infty)\) such that \(x_2 > x_1\)

[The proof is similar for the case \(x_1 > x_2\)]

Then from (2.1) we have

\[ \left| y(x_2) - y(x_1) \right| = \left| \int_a^{x_1} (x_1 - t)^{\alpha-1} f(t) dt - \int_a^{x_2} (x_2 - t)^{\alpha-1} f(t) dt \right| \]

From (2.1) we have

\[ \left| y(x_2) - y(x_1) \right| = \lim_{\delta \rightarrow 0} \left| \int_a^{x_1} ((x_1 - t)^{\alpha-1} - (x_1 - t)^{\alpha-1}) f(t) dt + \int_a^{x_2} (x_2 - t)^{\alpha-1} f(t) dt \right| \]

Since \(f\) is bounded by \(M\) and \(0 < \alpha \leq 1\), \(x_2 > x_1\) it follows that

\[ \left| y(x_2) - y(x_1) \right| = \lim_{\delta \rightarrow 0} \left| \int_a^{x_1} ((x_1 - t)^{\alpha-1} - (x_1 - t)^{\alpha-1}) f(t) dt + \int_a^{x_2} (x_2 - t)^{\alpha-1} f(t) dt \right| \]

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\[ = \lim_{\delta \to 0} \int_0^{x_2-\delta} (x_1-t)^{\alpha-1} - (x_2-t)^{\alpha-1} dt \]

\[ \leq \lim_{\delta \to 0} \int_a^{x_2-\delta} M \left\{ (x_2-t)^{\alpha-1} - (x_1-t)^{\alpha-1} \right\} dt + M \lim_{\delta \to 0} \int_{x_2-\delta}^{x_2} (x_2-t)^{\alpha-1} dt \]

\[ + \lim_{\delta \to 0} \int_{x_2-\delta}^{x_2} M(x_2-t)^{\alpha-1} dt \]

\[ \leq M \lim_{\delta \to 0} \left\{ \frac{-(x_1-t)^{\alpha}}{\alpha} \right\}^{x_2-\delta}_{x_2} - \frac{-(x_2-t)^{\alpha}}{\alpha} \right\}^{x_2-\delta}_a + M \frac{(x_2-a)^{\alpha}}{\alpha} + M \frac{(x_2-x_1)^{\alpha}}{\alpha} \]

\[ \leq 2M \frac{(x_2-x_1)^{\alpha}}{\alpha}, \text{ because} \]

\[ M \frac{(x_1-a)^{\alpha}}{\alpha} < M \frac{(x_2-a)^{\alpha}}{\alpha} \]

Let \( \delta \) be a positive number such that \( x_2-x_1 < \delta \), it follows

\[ |y(x_2) - y(x_1)| < \frac{2M\delta^{\alpha}}{\alpha} \]

Now let \( \varepsilon \) be any positive number and choose \( \delta \) such that

\[ \delta = \left( \frac{\alpha \varepsilon}{2M} \right)^{\frac{1}{\alpha}} \]

Then we have

\[ |y(x_2) - y(x_1)| < \varepsilon \quad \text{ when ever } |x_2 - x_1| < \delta \]

This shows that \( y(x) \) is continuous on \([a, \infty)\).

Since \( h_\alpha(x) = g_\alpha(x) + y(x), (x \in (a, \infty)) \) therefore \( h_\alpha \) is continuous on \((a, \infty)\).

**THEOREM OF DERIVATIVE**

If \( 0 < \alpha \leq 1 \) and \( h_\alpha(x) \) is continuous on \((a,b)\),

\[ |h_\alpha(x)| \leq M \text{ for all } x \in (a,b), \text{ where } (M \in \mathbb{R}^+, M > 0). \]

Then

\[ \int_a^x I^\alpha_a h_\alpha(x) = h_\alpha(x) \quad \text{ for all } x \in (a,b), \] where \( I^\alpha_a \) and \( I^\alpha_a \) are define by [8].

**Proof**

Since \( h_\alpha(x) \) is continuous and bounded by (theorem2).

\[ I^\alpha_a h_\alpha(x) \text{ exists for } a \leq x \leq b. \]

Now since \( 0 < \alpha \leq 1 \) take \( n = 1 \) then by [8] p(11) we have

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\[
\int_a^x \left( \frac{dx}{\alpha} \right) = \frac{1}{\alpha} \int_a^x \left( \frac{dx}{\alpha} \right)
\]
and from [8] p(17) we get

\[
\int_a^x h_\alpha = D \int_a^x h_\alpha
\]

for all \(x \in (a, b]\)

CONCLUSIONS
In this paper we proved (theorem1) of continuity and (theorem2) of derivative. In the future we will use them to find the solution of certain kind of fractional differential equation.

REFERENCES
[8] AL-Taba Kehaly, M. A., on the existence and uniqueness for fractional differential equation MSC. The sis, University of Musol.