Numerical Solution of Linear Delay Fredholm Integral Equations by using Homotopy Perturbation Method

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ABSTRACT
The main purpose of this work is to propose the Homotopy perturbation method for solving first type Linear Delay Fredholm Integral Equations (LDFIE). A Comparison between the numerical solution and exact solution has been made depending on least squares errors. Results are presented in tables and figures. Efficiency and accuracy are appeared in applying of this the proposed method.

1-INTRODUCTION
Delay integral equation is an equation in which the unknown function appears under an integral sign. The Delay integral lies in ability to describe processes with retarded time. The importance of these equations in various branch of Technology, economics, biology, has been recognized recently and tersest. Many authors have studied the delay integral equations. For example [Burton T.], [Kurger R.], [AL-Shakhaly T.] have studied some basic concepts and facts about the Delay integral equations.

This work can classify the linear Fredholm delay integral equations as:-
1-The first type, when the delay appears in the unknown function \( u(x) \) Inside the integral sign such that:

\[
h(x)u(x) = f(x) + \lambda \int_{a}^{b} k(x,t)u(t-\tau)dt \quad .... (1)
\]

2-The second type, when the delay appears outside the integral sign such that:
\[ h(x)u(x - \tau) = f(x) + \lambda \int_a^b k(x, t)u(t)dt \quad \ldots \quad (2) \]

3- The third type, when the two delays appear in the unknown function \( u(x) \) inside and outside the integral sign such that:

\[ h(x)u(x - \tau_1) = f(x) + \lambda \int_a^b k(x, t)u(t - \tau_2)dt \quad \ldots \quad (3) \]

Where \( h, f \) and \( k \) are known function. The function \( f \) is said to be the driving term and \( k \) is said to be kernel function that depends on \( x, t \) and \( \lambda \) is scalar parameter (in this work \( \lambda = 1 \), \( a \) and \( b \) known constants, 
\( \tau, \tau_1, \tau_2 \) are positive constant numbers and \( u \) is the unknown function that must be determined.

If \( h(x) = 0 \), the equation above are called Fredholm delay integral equation of the first kind otherwise when \( (h(x) = 1) \) it's called Fredholm delay integral equation of the second kind. [1, 2, 8]

THE HOMOTOPY PERTURBATION METHOD

The Homotopy perturbation method was first proposed by Chinese mathematician He J. Huan for solving integral equations, linear and nonlinear problems has been developed by scientists and engineers, becomes this method continuously deforms the difficult equation under study into a simple equation, easy to solve [10]. In this method the solution is considered as the summation of an infinite series, which usually converges rapidly to the exact solution [13].

SOME BASIC CONCEPTS OF THE HOMOTOPY PERTURBATION METHOD

We give some basic concepts of the Homotopy perturbation method, we recall the following definitions:

**Definition:** [13]

Let \( X \) and \( Y \) be two topological spaces. If \( f \) and \( g \) are continuous maps of the space \( X \to Y \), it is said that \( f \) is Homotopic to \( g \), if there exists a continuous map \( H : X \times [0,1] \to Y \), such that:

\[ H(x,0) = f(x), \quad H(x,1) = g(x) \quad \forall x \in X \]

Then the map is called Homotopy between \( f \) and \( g \).

**Theorem** [3]

The Homotopy relation is an equivalence relation.
Definition:
Let $X$ and $Y$ be two topological spaces. Two continuous functions $f$ and $g$ are said Homotopic relative to $A \subseteq X$ if there exists a continuous function $H : X \times [0,1] \rightarrow Y$ such that:

$$
H(x,0) = f(x) \quad \forall x \in X
$$

$$
H(x,1) = g(x)
$$

$$
H(a,p) = f(a) = g(a), \forall p \in [0,1], \forall a \in A
$$

To illustrate the basic idea of the homotopy perturbation method, we considering the following integral equation:

$$
\Omega \in = x f u A, 0 ) ( \ldots (4)
$$

Where $A$ is a general integral operator, $f$ is a known function of $x$, $\Omega = \{(x,t) \mid a \leq x, t \leq b\}$ and $u$ is the unknown function that must be determined.

The operator $A$, generally speaking, is divided into two parts $L$ and $N$, where $L$ is linear, and $N$ is nonlinear. Therefore equation (4) can be rewritten as follows:

$$
L(u) + N(u) - f(x) = 0
$$

By the homotopy technique [4, 5, 9, 11, 12], we construct a homotopy $v(x,p) : \Omega \times [0,1] \rightarrow R$ which satisfies the homotopy equation:

$$
H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(x)] = 0
$$

Or

$$
H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(x)] = 0 \ldots (5)
$$

Where $p \in [0,1]$ and $u_0$ is an initial approximation of the solution of equation (4). Obviously from equation (5) we have

$$
H(v,0) = L(v) - L(u_0) = 0
$$

$$
H(v,1) = A(v) - f(x) = 0
$$

The changing process of $p$ from zero to unity is just that of $v(x,p)$ from $u_0(x)$ to $u(x)$.

In topology, these is called deformation, $L(v) - L(u_0)$, and $A(v) - f(x)$ are homotopic.

Therefore $L(v) - L(u_0) \equiv A(v) - f(x), \quad x \in \Omega$

Assume that the solution of equation (5) can be written as a power series in $p$ as follows:

$$
v(x) = v_0(x) + pv_1(x) + p^2v_2(x) + \ldots \ldots (6)
$$

By setting $p = 1$ in equation (6), one can get:

$$
u(x) = \lim_{p \rightarrow 1} v(x) = v_0(x) + v_1(x) + v_2(x) + \ldots \ldots (7)
$$

This is the solution of equation (4).
The series (7) is convergent for some cases. However, the convergent rate depends on $A(v)$ [6, 7].

THE HOMOTOPY PERTURBATION METHOD FOR SOLVING LINEAR DELAY INTEGRAL EQUATIONS

In this work the homotopy perturbation is employed for solving linear delay integral equations. Consider the first type linear delay Fredholm integral equation:

$$u(x) = f(x) + \lambda \int_{a}^{b} k(x,t)u(t-\tau)dt \quad x \in [a,b] \quad \ldots (8)$$

we rewrite equation (8) as

$$A(u) - f(x) = 0$$

where $A(u) = u(x) - \lambda \int_{a}^{b} k(x,t)u(t-\tau)dt$

Then the integral operator $A$ can be divided into two parts $L$ and $N$ such that equation (8) becomes:

$$L(u) + N(u) - f(x) = 0$$

where $L(u) = u$ and $N(u) = -\lambda \int_{a}^{b} k(x,t)u(t-\tau)dt$

By using the homotopy technique, we can construct a homotopy $v(x, p) : [a, b] \times [0,1] \rightarrow R$ which satisfies:

$$H(v, p) = (1, p)[v(x, p) - u_0(x)] + p[v(x, p) - \lambda \int_{a}^{b} k(x,t)v(t-\tau, p)dt - f(x)] = 0 \quad \ldots (9)$$

where $p \in [0,1]$ and $u_0$ is the initial approximating to the solution of equation (8).

By using equation (9) it follows that

$$H(v,0) = v(x,0) - u_0(x) = 0$$

$$H(v,1) = v(x,1) - \lambda \int_{a}^{b} k(x,t)v(t-\tau, 1)dt - f(x) = 0$$

and the changing process of $p$ from zero to unity is just that of $v(x, p)$ from $u_0(x)$ to $u(x)$. In topology, this called Homotopic.

Therefore $v(x,0) - u_0(x) \equiv v(x,1) - \lambda \int_{a}^{b} k(x,t)v(t-\tau, 1)dt - f(x) \quad x \in [a,b]$

Next, we assume that the solution of equation (9) can be expressed as

$$v(x, p) = \sum_{i=0}^{\infty} p^i v_i(x) \quad \ldots (10)$$

Therefore the approximated solution of the integral equation (8) can be obtained as follows:
By substituting the approximated solution given by equation (10) into equation (9) one can get
\[ \sum_{i=0}^{\infty} p^i v_i(x) - u_0(x) + pu_0(x) + p{-\lambda \int_{a}^{b} k(x,t) \sum_{i=0}^{\infty} p^i (t-\tau) dt} - f(x) = 0 \]

Then by equating the terms with identical power of \( p \) one can have
\[ \begin{align*}
  p^0 : v_0(x) - u_0(x) &= 0 \quad \ldots (12.a) \\
  p^1 : v_1(x) + u_0(x) - f(x) - \lambda \int_{a}^{b} k(x,t) v_0(t-\tau) dt &= 0 \quad \ldots (12.b) \\
  p^j : v_j(x) - \lambda \int_{a}^{b} k(x,t) v_{j-1}(t-\tau) dt &= 0 \quad j = 1,2,3,\ldots \quad \ldots (12.c)
\end{align*} \]

For simplicity we set \( v_0(x) = u_0(x) = f(x) \), and then equation (12.a) is automatically satisfied. By substituting \( v_0(x) = u_0(x) = f(x) \) into equation (12.b) one can get
\[ v_1(x) = \lambda \int_{a}^{b} k(x,t) v_0(t-\tau) dt \]

and in general we have:
\[ v_j(x) = \lambda \int_{a}^{b} k(x,t) v_{j-1}(t-\tau) dt \quad j = 1,2,3,\ldots \]

Then by substituting \( v_i(x) \), \( i = 0,1,2,\ldots \) into equation (11) we can get the approximated solution of the integral equation (8).[12,14]

**NUMERICAL EXAMPLES**

In this section we present two examples of the linear Fredholm delay integral differential equation that are solved by the Homotopy perturbation method (HPM).

**Example (1)**
Consider the first type Linear Fredholm Delay Integral Equation:
\[ u(x) = x^2 + \frac{5}{3} x + \int_{-1}^{1} xt u(t-1) dt \]

Where \( a = -1, b = 1 \), \( f(x) = x^2 + \frac{5}{3} x \), and \( k(x,t) = x t \), with the exact solution \( u(x) = x^2 + x \).

To solve this example via the homotopy perturbation method, the algorithm start with initial approximation \( v_0(x) = f(x) \) to obtain a first approximation
\[ v_0(x) = u_0(x) = f(x) = x^2 + \frac{5}{3} x \].
Then

\[ v_1(x) = \int_{-1}^{1} k(x,t) v_0(t-1) dt = \int_{-1}^{1} xt \left( (t-1)^2 + \frac{5}{3} (t-1) \right) dt = -\frac{2}{9} x \]

In this case, let \( N=1 \), then

\[ u(x) \equiv \sum_{i=0}^{N} v_i(x) \Rightarrow u_1(x) = v_0(x) + v_1(x) = x^2 + \frac{13}{9} x = x^2 + 1.4444 x \]

Next, we must find \( v_2(x) \):

\[ v_2(x) = \int_{-1}^{1} k(x,t) v_1(t-1) dt = -\frac{4}{27} x \]

In the case of \( N=2 \),

\[ u(x) \equiv \sum_{i=0}^{N} v_i(x) \Rightarrow u_2(x) = v_0(x) + v_1(x) + v_2(x) = x^2 + \frac{35}{27} x = x^2 + 1.2963 x \]

Table (1) presents the comparison between the exact and approximated solution by using homotopy perturbation method for \( N=21 \) with least square error (L.S.E).

Figure (1) shows the approximated solution by using homotopy perturbation method and the exact solution.

Table (1): Numerical results for example (1)

<table>
<thead>
<tr>
<th>( x )</th>
<th>Exact solution</th>
<th>Approximated solution using homotopy perturbation method</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1100</td>
<td>0.1100</td>
</tr>
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<td>0.2</td>
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<td>0.2400</td>
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<td>0.7500</td>
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<tr>
<td></td>
<td>L.S.E</td>
<td>0.0000</td>
</tr>
</tbody>
</table>
Example (2):
Consider the first type of linear Fredholm delay integral differential equation:

\[
u(x) = 5 \cos(x) - \frac{629}{299} \cos(x) - \int_0^1 \sin(t) \cos(x) u(t-1) \, dt
\]

Where \(a = 0, b = 1\) and \(f(x) = 5 \cos(x) - \frac{629}{299} \cos(x)\), and

\[
k(x,t) = \sin(t) \cos(x), \text{ with the exact solution } u(x) = 5 \cos(x).
\]

To solve this example via the homotopy perturbation method, the algorithm starts with initial approximation \(v_0(x) = f(x)\) to obtain a first approximation

\[
v_0(x) = u_0(x) = f(x) = 5 \cos(x) - \frac{629}{299} \cos(x)
\]

Then

\[
v_1(x) = \int_0^1 k(x,t) v_0(t-1) \, dt = \int_0^1 \sin(t) \cos(t)(5 \cos(t-1) - \frac{629}{299} \cos(t-1)) \, dt = 1.21
\]

In this case, let \(N=1\), then

\[u(x) \approx \sum_{i=0}^{N} v_i(x)\]

\[u(x) = \sum_{i=0}^{N} v_i(x) \Rightarrow u_1(x) = v_0(x) + v_1(x) = 4.1149 \cos(x)\]

Next, we must find \(v_2(x)\):
\[ v_2(x) = \frac{1}{0} k(x, t) v_1(t-1)dt = 0.5127\cos(x) \]

In the case of \( N=2 \),

\[ u(x) \equiv \sum_{i=0}^{N} v_i(x) \Rightarrow u_2(x) = v_0(x) + v_1(x) + v_2(x) = 4.6276\cos(x) \]

Table (2) presents the comparison between the exact and approximated solution by using homotopy perturbation method for \( N=15 \) with least square error (L.S.E).

Figure (2) shows the approximated solution by using homotopy perturbation method and the exact solution.

Table (2): Numerical results for example (2)

<table>
<thead>
<tr>
<th>( x )</th>
<th><strong>Exact solution</strong></th>
<th><strong>Approximated solution using homotopy perturbation method</strong></th>
</tr>
</thead>
<tbody>
<tr>
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<td>4.1267</td>
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<tr>
<td>0.8</td>
<td>3.4835</td>
<td>3.4835</td>
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<tr>
<td>0.9</td>
<td>3.1080</td>
<td>3.1080</td>
</tr>
<tr>
<td>1</td>
<td>2.710</td>
<td>2.710</td>
</tr>
</tbody>
</table>

L.S.E 0.0000
CONCLUSIONS
In this paper, we used Homotopy Perturbation Method for solving first type of linear delay Fredholm integral equation. The results show a marked improvement in the least square error (L.S.E).

From the present study, we can conclude the following points:
1- The Homotopy perturbation method gives more accurate results when the solutions of the problems are polynomials.
2- The good approximation depend on $N$, as $N$ increased, the error term is decreased.
3- The Homotopy perturbation method for solving any linear integral equation required the initial approximation; in this work we suppose the initial approximation $u_0(x) = f(x)$

REFERENCES


