The Solvability, Controllability & Observability of Infinite Dimensional Nonlinear Control System Using Banach Fixed Point Theorem

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Abstract

In this paper, the theoretical results of the solvability, controllability and observability of the mild solution to the following nonlinear dynamic control system

$$\frac{d u_w(t)}{dt} + A u_w(t) = f(t, u_w(t)) + \int_{s=0}^{s=t} h(t-s) g(s, u_w(s)) ds + (B w)(t)$$

$$u_w(0) = u_0$$

have been discussed and proved via Banach fixed point theorem and strongly continuous semigroup theory.

Keywords

Controllability, Observability, Fixed point theorem, Control systems and semigroup theory.

AMS Classification

93Bxx , 47H10 , 93Cxx , 47D06.

قابلية الحل, قابلية السيطرة وقابلية الملاحظة لنظام سيطرة لاخطي ذو بعد لانهائي بأستخدام نظرية النقطة الثابتة باناخ

الخلاصة

في هذا البحث, تم مناقشة و برهنة النتائج النظرية لقابلية الحل, قابلية السيطرة وقابلية الملاحظة للحل المعتدل لنظام سيطرة ديناميي لاخطي والمعرف كالآتي :

$$\frac{d u_w(t)}{dt} + A u_w(t) = f(t, u_w(t)) + \int_{s=0}^{s=t} h(t-s) g(s, u_w(s)) ds + (B w)(t)$$

$$u_w(0) = u_0$$

باستخدام نظرية النقطة الثابتة باناخ ونظرية شبه الزمرة المستمرة بقوة.

1. Introduction

A well developed theory of controllability and observability for linear system have been available for many years , even in infinite dimension space [1]. The controllability and observability in nonlinear control system have been limited and any success depends upon particular classes of nonlinearity. The choice of the appropriate method depends on the type of nonlinearity in the state equation. There are different methods for investigation of observability and controllability for different types of nonlinear system, fixed point theorems for non-linear mappings [2], theory of vector field and Lie algebras [3], perturbation method [4], and Maximum principle [5]. In this work, Banach fixed point theorem has been adapting. There are various fixed-point theorems available, the most popular being Schauder's fixed point theorem [6],

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Banach fixed point theorem [7] and Darboux fixed point theorem [8]. Fixed point theorems have found a wide applications in both , the theory and numerical aspects of differential equations.

In recent years, Fazal and Khan [1] have studied the controllability and observability of the mild solution to the nonlinear system:

$$\frac{d\,u(t)}{dt} = A\,u(t) + N\,u(t)$$
$$y(t) = C\,u(t)$$
$$u(0) = u_0 \qquad \ldots(1)$$

where A is the infinitesimal generator of the strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on a reflexive complex Banach space X into X, the function y(·) is referred to as the output which is belong to Y , a reflexive complex Banach space of the outputs , C is bounded linear operator defined from X into Y and N is non-linear operator defined on X.

A continuous function $u(\cdot)\in C(J_0 : X)$ is said to be the mild solution of the nonlinear system defined in (1) if u(·) satisfies the following form:

$$u(t) = T(t)\,u_0 + \int_{s=0}^{s=t} T(t-s)\,N\,u(s)\,ds \qquad \ldots(2)$$

where $C(J_0 : X)$ stands for the set of all continuous functions defined from J_0 into X and $J_0 = [0,t_1]$. As well as in [1], some results concerning the controllability and observability of the mild solution defined in (2) to the nonlinear system defined in (1) are reproduced by using a new fixed point approach, although these results have already been established by Quinn and Carmichael [8] using Darboux fixed point theorem.

Bahuguna.D in 1997 [9], has studied the local existence of the mild solution to the semilinear initial value problem:

$$\frac{d\,u(t)}{dt} + A\,u(t) = f\big(t, u(t)\big) + \int_{s=0}^{s=t} h(t-s)\,g\big(s, u(s)\big)ds$$
$$u(0) = u_0 \qquad \ldots(3)$$

where A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ defined from D(A)⊂X into X , where X is suitable complex Banach space, f and g are a nonlinear continuous maps defined from $[0, t_1]\times X$ into X, h is the real valued continuous function defined from $[0, t_1]$ into $\Re$ where $\Re$ is the set of all real numbers .

A continuous function $u(\cdot)\in C(J_0 : X)$ is said to be a mild solution for the nonlinear system defined in (3) if u(·) satisfies the following form:

$$u(t) = T(t)\,u_0 + \int_{s=0}^{s=t} T(t-s)\Big[f\big(s, u(s)\big) + \int_{\tau=0}^{\tau=s} h(t-s)\,g\big(\tau, u(\tau)\big)\,d\tau \Big] ds \qquad \ldots(4)$$

,where $C(J_0 : X)$ stands for the set of all continuous functions defined from J_0 into X and u(·) is a continuous function belong to $C(J_0 : X)$.

Radhi and Manaf [10] have studied the solvability and exactly controllability of the mild solution to the following nonlinear control system:

$$\frac{d\,u_w(t)}{dt} + A\,u_w(t) = f\big(t, u_w(t)\big) + \int_{s=0}^{s=t} h(t-s)\, g\big(s, u_w(s)\big)\,ds + (B\,w)(t),\ t > 0$$
$$u(0) = u_0 \qquad \ldots(5)$$

where A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ defined from D(A)⊂X into X , f and g are a nonlinear continuous maps defined from $[0, t_1] \times X$ into X, h is the real valued continuous function defined from $[0, t_1]$ into $\Re$ and the control function $w(.)$ belong to $L^2(J_0, O)$, a Banach space of admissible control functions with O as a Banach space (control space) and $J_0 = [0, t_1]$, B is a bounded linear operator from O into X.

A continuous function $u_w(\cdot) \in C(J_0 : X)$ is said to be a mild solution to the nonlinear control system defined in (5) if $u_w(\cdot)$ satisfies the following form:

$$u_w(t) = T(t)\,u_0 + \int_{s=0}^{s=t} T(t-s)\Big[f\big(s, u_w(s)\big) + \int_{\tau=0}^{\tau=s} h(t-s)\, g\big(\tau, u_w(\tau)\big)\,d\tau + B\,w(s) \Big]\,ds \qquad \ldots(6)$$

For every control function $w(.) \in L^2(J_0, O)$.

2. Preliminaries

Definition 2.1 [11]

Let $\{T(t)\}_{t\geq 0}$ be a family of bounded linear operators on a complex Banach space X. The infinitesimal generator A of T(t) is the linear operator defined by:

$$A(x) = \lim_{h\to 0} \frac{T(h)x - x}{h}, \quad \text{for}$$

$x \in D(A)$, where

$$D(A) = \left\{ x \in X : \lim_{h\to 0} \frac{T(h)x - x}{h} \ \text{exists} \right\}$$

The following is the most important theorem in the theory of semigroup and its applications to linear unbounded operator.

Theorem (Hill-Yosida) 2.1 [11]

A linear unbounded operator A is the infinitesimal generator of a C_0 (strongly continuous semigroup) of contractions T(t), $t \geq 0$ if and only if

A is closed and the closure of the domain A is equal to X.

The resolvent set ρ(A) of A contains R^+ and for every $\lambda > 0$, $\|R(\lambda; A)\| \leq \frac{1}{\lambda}$,

where R^+ is the set of all positive real numbers.

Diagram 2.1 [12]

The diagram (1) represents the relations among a semigroup, its generator and its Resolvent , see diagram (1).

Definition 2.2 [11]

Given any two points $u_0, u_\gamma \in X$, where X is a complex Banach space. The mild solution defined in (6) to the nonlinear control system defined in (5) is said to be exactly controllable on $J_0^* = [0, \gamma]$, if there exist a control $\underline{w}(\cdot) \in L^2(J_0^*, O)$ such that the mild solution $u_{\underline{w}}(\cdot)$ defined in (6) satisfy the following conditions $u_{\underline{w}}(0) = u_0$ and $u_{\underline{w}}(\gamma) = v_0$, where $u_{\underline{w}}(\cdot)$ is a continuous function depend on the control function $\underline{w}(\cdot)$ which is belong to $L^2(J_0^*, O)$.

3. The Solvability and Controllability of Nonlinear Dynamic Control System

In the present work, we have studied the solvability and exactly controllability of the mild defined in (6) to the nonlinear control system defined in (5) are reproduced by using Banach fixed point theorem and strongly continuous semigroup theory , although these results have already been established by Radhi and Manaf [10] by using Schauder fixed point theorem and compact strongly continuous semigroup theory. The work become stronger than [10] by attenuation of some conditions in it, i.e. we omitted compact semigroup and some other conditions that are used and we replaced them by strongly continuous semigroup theory.

Main Result 3.1

To study the solvability and controllability of the mild solution defined in (6) of the nonlinear control system defined in (5), the following hypotheses are adopted in the present work and as follow:

$(\mathbf{A_1})$ A be the infinitesimal generator strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ and A defined from D(A) ⊂ X into X , where X is a complex Banach space.

$(\mathbf{A_2})$ Let ρ' be a positive constant such that $\rho > \rho'$ satisfy the condition $\|T(t)u_0 - u_0\|_X \leq \rho'$, $0 \leq t \leq t_1$.

$(\mathbf{A_3})$ Let f and g are nonlinear continuous maps defined from $J_0 \times X$ into X satisfy the following conditions:

$(\mathbf{A_3} \cdot \mathrm{i})$ $\|f(t,v)\|_X \leq N_1$, $\|g(t,v)\|_X \leq N_2$, $\forall v \in X$ and $t \in J_0 = [0, t_1]$, N_1 and N_2 are positive constants.

$(\mathbf{A_3} \cdot \mathrm{ii})$

$\|f(t,v_1) - f(t,v_2)\|_X \leq L_0\|v_1 - v_2\|_X$,

$\|g(t,v_1) - g(t,v_2)\|_X \leq L_1\|v_1 - v_2\|_X$

$\forall v_1, v_2 \in X; 0 \leq t \leq t_1$

, where $L_0 > 0$ and $L_1 > 0$.

$(\mathbf{A_4})$ Let h be the real valued continuous function defined from J_0 into $\Re$ where $\Re$ is the set of all real numbers with the positive constant

$$h_{t_1} = \int_{s=0}^{s=t_1} |h(s)|\, ds .$$

$(\mathbf{A_5})$ Let $w(.)$ be an arbitrary control function belong to $L^2(J_0, O)$, a Banach space of admissible control functions with O as a Banach space (control space).

$(\mathbf{A_5}.\mathrm{i})$ The bounded linear operator B defined from O into X, i.e. $\|Bv\|_X \leq K_0\|v\|_O, \forall v \in O$, K_0 is a positive constant .

$(\mathbf{A_5}.\mathrm{ii})$ There exist a positive constant k_1 such that $\|w(t)\|_O \leq K_1$, for all $0 \leq t \leq t_1$.

$(\mathbf{A_6})$ If $t_1 \in R^+$, {where R^+ the set of all positive real numbers}, satisfy the following conditions:

$(\mathbf{A_6}.\mathrm{i})$

$$t_1 \leq \frac{\rho - \rho'}{M\left(N_1 + h_{t_1}N_2 + K_0K_1\right)},$$

where $\rho, \rho', N_1, N_2, h_{t_1}, K_0, K_1$ and M are positive real numbers.

$(\mathbf{A_6}.\mathrm{ii})$ $$t_1 \leq \frac{\left(1 - \frac{\rho}{\rho + 1}\right)}{M\left(L_0 + h_{t_1}L_1\right)},$$

where ρ, L_0, L_1, h_{t_1} and M are positive real numbers.

$(\mathbf{A_7})$ The linear operator G from $L^2(J_0^*, O)$ into X defined by:

$G\,w = \int_{s=0}^{s=\gamma} T(\gamma - s)B\,w(s)\,ds \quad , \forall$ $w(.) \in L^2(J_0^*, O)$. Induces an invertible operator $\tilde{G}$ defined from $L^2(J_0^*, O)/\mathrm{Ker}\,G$ into X, where is quotient space, where $J_0^* = [0, \gamma]$.

$(\mathbf{A_8})$ There exist a positive constant I_0 such that $\|\tilde{G}^{-1}\| \le I_0$.

$(\mathbf{A_9})$ Let $0 < \gamma < t_1$ and satisfy the following conditions:

$(\mathbf{A_9}.i)$

$$\rho = \rho' + M\,N_1\,\gamma + M\,h_\gamma\,N_2\,\gamma + M\,K_0\,I_0\left[\|v_0\|_X - M\,\|u_0\|_X - M\,(N_1 + h_\gamma N_2)\,\gamma\right]\gamma$$

$(\mathbf{A_9}.ii)$

$$\gamma \le \left[\frac{\left(1 - \frac{\rho'}{\rho}\right)}{M\,L_0 + M\,h_\gamma L_1 + M^2 K_0 I_0 (L_0 + h_\gamma L_1)}\right]^{\frac{1}{2}}$$

Remark 3.1

The condition $(\mathbf{A_7})$ in our assumptions can be satisfied see appendix in [14].

Lemma 3.1

Consider the nonlinear control system defined in (5). Let $Y = C(J_0 : X)$, where Y is a complex Banach space with the sup–norm defined as follow: $\|y\|_Y = \sup_{0 \le t \le t_1} \|y(t)\|_X$ and assume S_w be a nonempty subset of Y which is depend on the control function $w(.) \in L^2(J_0, O)$, define as follow:

$$S_w = \{u_w \in Y : u_w(0) = u_0,\ \|u_w(t) - u_0\|_X \le \rho; 0 \le t \le t_1\}$$

Then S_w is a closed subset of Y for every control function $w(.) \in L^2(J_0, O)$

Proof

Let the sequence $u_w^n \in S_w$ such that $u_w^n \xrightarrow{P.C} u_w$ as $n \to \infty$, where u_w^n is a sequence of continuous function depending on the control function $w(.) \in L^2(J_0, O)$ which is point wise convergent to u_w, to prove that $u_w \in S_w$, i.e. to prove the following steps:

Step (1) proves that $u_w \in Y$, $\forall$ control function $w(.) \in L^2(J_0, O)$.

Step (2) proves that $u_w(0) = u_0$, $\forall$ control function $w(.) \in L^2(J_0, O)$.

Steps (3) show that $\|u_w(t) - u_0\|_X \le \rho$, $\forall$ control function $w(.) \in L^2(J_0, O)$.

For the proof of step (1), since $u_w^n \in S_w$ depending on the continuity property with the point wise convergent of the sequence u_w^n, we have that the sequence u_w^n is uniformly convergent to u_w, depending on the uniformly convergent property, we have that $u_w \in Y$, $\forall$ $w(.) \in L^2(J_0, O)$, so step (1) holds. To proof step (2), since the sequence u_w^n is uniform convergent to u_w and

$$\|u_w^n - u_w\|_Y = \sup_{0 \le t \le t_1} \|u_w^n(t) - u_w(t)\|_X \quad ,$$

where Y is a complex Banach space then $\sup_{0 \le t \le t_1} \|u_w^n(t) - u_w(t)\|_X \to 0$ as $n \to \infty$, which implies that $\|u_w^n(t) - u_w(t)\|_X \to 0$ as $n \to \infty$, $\forall\, 0 \le t \le t_1$ i.e. $\lim_{n\to\infty} u_w^n(t) = u_w(t)$, $\forall\, 0 \le t \le t_1$.

As a special case when t=0, one can get $\lim_{n\to\infty} u_w^n(0) = u_w(0)$, $\lim_{n\to\infty} u_0 = u_w(0)$ implies that $u_0 = u_w(0)$, $\forall$ control function $w(.) \in L^2(J_0, O)$.

To complete the proof, the validation of step (3) is needed as follow:

$\|u_w(t)-u_0\|_X = \left\|\lim_{n\to\infty} u_w^n(t)-u_0\right\|_X =$

$\left\|\lim_{n\to\infty} u_w^n(t)-\lim_{n\to\infty} u_0\right\|_X = \lim_{n\to\infty}\left\|u_w^n(t)-u_0\right\|_X$

$\leq \lim_{n\to\infty}\rho = \rho$ .Hence $\|u_w(t)-u_0\|_X \leq \rho$,

$\forall$control function $w(.)\in L^2(J_0, O)$.

We conclude that S_w is a closed subset of Y for every control function $w(.)\in L^2(J_0, O)$.

Theorem 3.1

Assuming that the hypotheses $(\mathbf{A_1})$ to $(\mathbf{A_6})$ hold. Then the nonlinear control system defined in (6) has a unique fixed point $L^2(J_0^*, O)$, $\forall$ control function $w(.)\in L^2(J_0, O)$.

Proof

For arbitrary control function $w(.)$ belong to $L^2(J_0, O)$, define the nonlinear map $F_w : S_w \to Y$ by:

$$(F_w u_w)(t) = T(t)u_0 + \int_{s=0}^{s=t} T(t-s)\left[f(s, u_w(s)) + \int_{\tau=0}^{\tau=s} h(s-\tau)\, g(\tau, u_w(\tau))d\tau + Bw(s) \right]ds \quad \ldots(7)$$

, $\forall$control function $w(.)\in L^2(J_0, O)$.

Our aim is then to prove that there exists a unique fixed point u_w of (7), i.e. there is a unique $u_w \in S_w$ such that $F_w u_w = u_w$ for arbitrary control function $w(.)\in L^2(J_0, O)$.

The Banach fixed point theorem is adapted to ensure the existence of a unique fixed point u_w of (7), for arbitrary control function $w(.)\in L^2(J_0, O)$ and as following steps:

Step (1) S_w is closed subset of Y for arbitrary control function $w(.)\in L^2(J_0, O)$.

Step (2) $F_w(S_w) \subseteq S_w$ for arbitrary control function $w(.)\in L^2(J_0, O)$.

Step (3) F_w is a strict contraction on S_w for arbitrary control function $w(.)\in L^2(J_0, O)$. By using lemma (3.1), step (1) holds. To prove step (2), let u_w be the arbitrary element in S_w such that $F_w u_w \in F_w(S_w)$, to show that $F_w u_w \in S_w$, the following are needed (see the definition of the set S_w in lemma 3.1).

1. $(F_w u_w)(0) = u_0$, $\forall\, w(.)\in L^2(J_0, O)$
2. $\|u_w(t)-u_0\|_X \leq \rho$, $\forall\, w(.)\in L^2(J_0, O)$, $0 \leq t \leq t_1$.

From the definition of the map F_w which is defined in (7), it is clear that (1) holds. To show (2) notices that:

$$\left\|(F_w u_w)(t)-u_0\right\|_X = \Bigg\| T(t)u_0 - u_0 + \int_{s=0}^{s=t} T(t-s)\Big[f(s,u_w(s)) + \int_{\tau=0}^{\tau=s} h(s-\tau)\, g(\tau,u_w(\tau))d\tau + Bw(s) \Big]ds \Bigg\|_X$$

$$\left\|(F_w u_w)(t)-u_0\right\|_X \le \left\|T(t)u_0-u_0\right\|_X + \int_{s=0}^{s=t}\left\|T(t-s) f(s,u_w(s))\right\|_X ds + \int_{s=0}^{s=t}\left\|T(t-s) Bw(s)\right\|_X ds + \int_{s=0}^{s=t}\left\|T(t-s)\int_{\tau=0}^{\tau=s} h(s-\tau)\, g(\tau,u_w(\tau))d\tau\right\|_X ds$$

Since T(t) is bounded linear operator for $0 \le t \le t_1$, there exists M>0 such that

$$\|T(t)x\|_X \le M\|x\|_X, \forall x \in X \qquad \ldots(8)$$

$$\left\|(F_w u_w)(t)-u_0\right\|_X \le \left\|T(t)u_0-u_0\right\|_X + M\int_{s=0}^{s=t}\left\|f(s,u_w(s))\right\|_X ds + M\int_{s=0}^{s=t}\left\|Bw(s)\right\|_X ds + M\int_{s=0}^{s=t}\int_{\tau=0}^{\tau=s}|h(s-\tau)| \left\|g(\tau,u_w(\tau))\right\|_X d\tau\, ds$$

By using the conditions $(\mathbf{A}_2)$, $(\mathbf{A}_4)$, $(\mathbf{A}_3 \cdot i)$, and $(\mathbf{A}_5)$, one can get:

$$\left\|(F_w u_w)(t)-u_0\right\|_X \le \rho' + M N_1 t + M K_0 K_1 t + M h_{t_1} N_2 t$$

$$\left\|(F_w u_w)(t)-u_0\right\|_X \le \rho' + \left(N_1 + K_0 K_1 + h_{t_1} N_2\right) M t, \ t \in [0, t_1]$$

$$\left\|(F_w u_w)(t)-u_0\right\|_X \le \rho' + \left(N_1 + K_0 K_1 + h_{t_1} N_2\right) M t_1 \qquad \ldots(9)$$

By using the condition $(\mathbf{A}_6 \cdot i)$ and the equation defined in (9), one can get:

$\left\|(F_w u_w)(t)-u_0\right\|_X \le \rho$, $\forall\, 0 \le t \le t_1$. Hence $F_w u_w \in S_w$, i.e. $F_w : S_w \to S_w$, $\forall$ control function $w(.) \in L^2(J_0, O)$.

To complete the proof, the validation of step (3) is needed and as follow:

Let $\overline{u}_w, \overline{\overline{u}}_w \in S_w$, where $\overline{u}_w, \overline{\overline{u}}_w$ are the continuous functions depend on The control function $w(.) \in L^2(J_0, O)$, then:

$$\left\|(F_w\overline{u}_w)(t)-(F_w\overline{\overline{u}}_w)(t)\right\|_X = \Bigg\| T(t)u_0 + \int_{s=0}^{s=t} T(t-s)\Big[f(s,\overline{u}_w(s)) + \int_{\tau=0}^{\tau=s} h(s-\tau)\, g(\tau,\overline{u}_w(\tau))\,d\tau + Bw(s)\Big]ds - T(t)u_0 - \int_{s=0}^{s=t} T(t-s)\Big[f(s,\overline{\overline{u}}_w(s)) + \int_{\tau=0}^{\tau=s} h(s-\tau)\, g(\tau,\overline{\overline{u}}_w(\tau))\,d\tau + Bw(s)\Big]ds \Bigg\|_X$$

$$\left\|(F_w\bar{u}_w)(t)-(F_w\bar{\bar{u}}_w)(t)\right\|_X \leq \left\|\int_{s=0}^{s=t} T(t-s)\Big[f(s,\bar{u}_w(s))-f(s,\bar{\bar{u}}_w(s))\Big]\right\|_X + \left\|\int_{s=0}^{s=t} T(t-s)\int_{\tau=0}^{\tau=s} h(s-\tau)\Big[g(\tau,\bar{u}_w(\tau))\,d\tau - g(\tau,\bar{\bar{u}}_w(\tau))\Big]d\tau\, ds\right\|_X \quad \ldots(10)$$

From the equations defined in (8) and(10) and using the condition $(\mathbf{A_4})$, we get:

$$\left\|(F_w\bar{u}_w)(t)-(F_w\bar{\bar{u}}_w)(t)\right\|_X \leq M\int_{s=0}^{s=t}\left\|f(s,\bar{u}_w(s))-f(s,\bar{\bar{u}}_w(s))\right\|_X ds + M\; h_{t_1}\int_{s=0}^{s=t}\left\|g(\tau,\bar{u}_w(\tau))-g(\tau,\bar{\bar{u}}_w(\tau))\right\|_X ds$$

By using the condition $(\mathbf{A_3}\cdot \text{ii})$, one can get:

$$\left\|(F_w\bar{u}_w)(t)-(F_w\bar{\bar{u}}_w)(t)\right\|_X \leq M\,L_0\left\|\bar{u}_w(s)-\bar{\bar{u}}_w(s)\right\|_X t + M\,h_{t_1}L_1\left\|\bar{u}_w(\tau)-\bar{\bar{u}}_w(\tau)\right\|_X t$$

$$\left\|(F_w\bar{u}_w)(t)-(F_w\bar{\bar{u}}_w)(t)\right\|_X \leq M\,L_0\sup_{0\leq t\leq t_1}\left\|\bar{u}_w(t)-\bar{\bar{u}}_w(t)\right\|_X t_1 + M\,h_{t_1}L_1\sup_{0\leq t\leq t_1}\left\|\bar{u}_w(t)-\bar{\bar{u}}_w(t)\right\|_X t_1$$

$$\left\|(F_w\bar{u}_w)(t)-(F_w\bar{\bar{u}}_w)(t)\right\|_X \leq \left(L_0+h_{t_1}L_1\right)M\,t_1\sup_{0\leq t\leq t_1}\left\|\bar{u}_w(t)-\bar{\bar{u}}_w(t)\right\|_X$$

By using the condition $(\mathbf{A_6}.\text{ii})$, we have:

$$\left\|(F_w\bar{u}_w)(t)-(F_w\bar{\bar{u}}_w)(t)\right\|_X \leq \left(1-\frac{\rho}{1+\rho}\right)\sup_{0\leq t\leq t_1}\left\|\bar{u}_w(t)-\bar{\bar{u}}_w(t)\right\|_X \quad \ldots(11)$$

Taking the sup-norm over $[0,t_1]$ to the equation defined in (11), one can get:

$$\sup_{0\leq t\leq t_1}\left\|(F_w\bar{u}_w)(t)-(F_w\bar{\bar{u}}_w)(t)\right\|_X \leq \left(1-\frac{\rho}{1+\rho}\right)\sup_{0\leq t\leq t_1}\left\|\bar{u}_w(t)-\bar{\bar{u}}_w(t)\right\|_X$$

, since $\|y\|_Y = \sup_{0\leq t\leq t_1}\|y(t)\|_X$, we obtain:

$$\left\|F_w\bar{u}_w - F_w\bar{\bar{u}}_w\right\|_Y \leq \left(1-\frac{\rho}{1+\rho}\right)\left\|\bar{u}_w-\bar{\bar{u}}_w\right\|_Y$$

,where $0<\left(1-\frac{\rho}{1+\rho}\right)<1$.

Thus F_w is a strict contraction map from S_w into S_w and therefore by the Banach fixed point theorem there is a unique $u_w\in S_w$ such that $F_w u_w = u_w$ for arbitrary control function $w(.)\in L^2(J_0,O)$.

Theorem 3.2

Assuming that the hypotheses $(\mathbf{A_1}),(\mathbf{A_2}),(\mathbf{A_3}),(\mathbf{A_4})$, $(\mathbf{A_5}\cdot \text{i}),(\mathbf{A_7}),(\mathbf{A_8})$ and $(\mathbf{A_9})$ are hold. Then the nonlinear control system defined in (6) is controllable on $J_0^* = [0,\gamma]$,where $0<\gamma<t_1$.

Proof

Using the condition $(\mathbf{A_7})$, for arbitrary function $u_{\underline{w}}(\cdot)$ define the control function $\underline{w}(\cdot)\in L^2(J_0^*,O)$ as follow:

$$\underline{w}(t) = \tilde{G}^{-1}\Big[v_0 - T(\gamma)u_0 - \int_{s=0}^{s=t} T(\gamma - s)\Big[f(s, u_{\underline{w}}(s)) + \int_{\tau=0}^{\tau=s} h(s-\tau)\, g(\tau, u_{\underline{w}}(\tau))\, d\tau \Big] ds \Big](t) \quad \ldots (12)$$

, where $u_{\underline{w}}(\cdot)$ is a continuous function depend on the control function $\underline{w}(\cdot) \in L^2(J_0^*, O)$.

Define a nonlinear map $F_{\underline{w}} : S_{\underline{w}} \to Y$ by:

$$(F_{\underline{w}} u_{\underline{w}})(t) = T(t)u_0 + \int_{s=0}^{s=t} T(t-s) \Big[f(s, u_{\underline{w}}(s)) + \int_{\tau=0}^{\tau=s} h(s-\tau)\, g(\tau, u_{\underline{w}}(\tau))\, d\tau + B\underline{w}(s) \Big] ds \quad \ldots(13)$$

Substitutes equation (12) in the equation (13), we have got:

$$(F_{\underline{w}} u_{\underline{w}})(t) = T(t)u_0 + \int_{s=0}^{s=t} T(t-s) \Big[f(s, u_{\underline{w}}(s)) + \int_{\tau=0}^{\tau=s} h(s-\tau)\, g(\tau, u_{\underline{w}}(\tau))\, d\tau \Big] ds + \int_{\theta=0}^{\theta=t} T(t-\theta)\, B\tilde{G}^{-1}\Big[v_0 - T(\gamma)u_0 - \int_{s=0}^{s=t} T(\gamma - s)\Big[f(s, u_{\underline{w}}(s)) + \int_{\tau=0}^{\tau=s} h(s-\tau) g(\tau, u_{\underline{w}}(\tau))\, d\tau \Big] ds \Big](\theta)\, d\theta \quad \ldots (14)$$

Our aim is then to prove that there exists a unique fixed point $u_{\underline{w}}$ of (14), i.e. there is a unique $u_{\underline{w}} \in S_{\underline{w}}$ such that $F_{\underline{w}} u_{\underline{w}} = u_{\underline{w}}$, for control function $\underline{w}(\cdot) \in L^2(J_0^*, O)$.

The Banach fixed point theorem is adapted to ensure the existence of a unique fixed point $u_{\underline{w}}$ of (14), for control function $\underline{w}(\cdot) \in L^2(J_0^*, O)$ and as following steps:

Step (1) $S_{\underline{w}}$ is closed subset of Y for control function $\underline{w}(\cdot) \in L^2(J_0^*, O)$.

Step (2) $F_{\underline{w}}(S_{\underline{w}}) \subseteq S_{\underline{w}}$ for control function $\underline{w}(\cdot) \in L^2(J_0^*, O)$.

Step (3) $F_{\underline{w}}$ is a strict contraction on $S_{\underline{w}}$ for control function $\underline{w}(\cdot) \in L^2(J_0^*, O)$. By using lemma (3.1), we have got $S_{\underline{w}}$ is closed subset of Y for arbitrary control $\underline{w}(\cdot) \in L^2(J_0^*, O)$, so for the special case $S_{\underline{w}}$ is also a closed subset of Y for control function $\underline{w}(\cdot) \in L^2(J_0^*, O)$.To prove step (2), let $u_{\underline{w}}$ be the arbitrary element in $S_{\underline{w}}$ such that $F_{\underline{w}} u_{\underline{w}} \in F_{\underline{w}}(S_{\underline{w}})$, to show that $F_{\underline{w}} u_{\underline{w}} \in S_{\underline{w}}$, the following are needed (see the definition of the set $S_{\underline{w}}$ in lemma 3.1).

1. $(F_{\underline{w}} u_{\underline{w}})(0) = u_0$, for control function $\underline{w}(\cdot) \in L^2(J_0^*, O)$.
2. $\|F_{\underline{w}} u_{\underline{w}}(t) - u_0\|_X \le \rho$, for control function $\underline{w}(\cdot) \in L^2(J_0^*, O)$.

, $0 \le t \le \gamma$. From the definition of the map $F_{\underline{w}}$ which is defined in (14), it is clear that (1) holds. To show (2) notice that:

$$\left\|(F_{\underline{w}} u_{\underline{w}})(t) - u_0\right\|_X = \Bigg\| T(t)u_0 + \int_{s=0}^{s=t} T(t-s)\Big[f\big(s, u_{\underline{w}}(s)\big) + \int_{\tau=0}^{\tau=s} h(s-\tau) g\big(\tau, u_{\underline{w}}(\tau)\big) d\tau \Big] ds + \int_{\theta=0}^{\theta=t} T(t-\theta) B \tilde{G}^{-1}\Big[v_0 - T(\gamma)u_0 - \int_{s=0}^{s=t} T(\gamma-s)\Big[f(s, u_{\underline{w}}(s)) + \int_{\tau=0}^{\tau=s} h(s-\tau) g(\tau, u_{\underline{w}}(\tau))\, d\tau \Big] ds \Big](\theta)\, d\theta \Bigg\|_X$$

$$\left\|(F_{\underline{w}} u_{\underline{w}})(t) - u_0\right\|_X \le \left\|T(t)u_0 - u_0\right\|_X + \int_{s=0}^{s=t} \Bigg\| T(t-s) \Big[f\big(s, u_{\underline{w}}(s)\big) + \int_{\tau=0}^{\tau=s} h(s-\tau) g\big(\tau, u_{\underline{w}}(\tau)\big) d\tau \Big] \Bigg\|_X ds + \int_{\theta=0}^{\theta=t} \Bigg\| T(t-\theta) B \tilde{G}^{-1}\Big[v_0 - T(\gamma)u_0 - \int_{s=0}^{s=t} T(\gamma-s)\Big[f(s, u_{\underline{w}}(s)) + \int_{\tau=0}^{\tau=s} h(s-\tau) g(\tau, u_{\underline{w}}(\tau))\, d\tau \Big] ds \Big](\theta) \Bigg\|_X d\theta$$

From the equation defined in (8), one can get:

$$\left\|(F_{\underline{w}} u_{\underline{w}})(t) - u_0\right\|_X \le \left\|T(t)u_0 - u_0\right\|_X + M \int_{s=0}^{s=t} \Bigg\| f\big(s, u_{\underline{w}}(s)\big) + \int_{\tau=0}^{\tau=s} h(s-\tau) g\big(\tau, u_{\underline{w}}(\tau)\big) d\tau \Bigg\|_X ds + M \int_{\theta=0}^{\theta=t} \Bigg\| B \tilde{G}^{-1}\Big[v_0 - T(\gamma)u_0 - \int_{s=0}^{s=t} T(\gamma-s)\Big[f(s, u_{\underline{w}}(s)) + \int_{\tau=0}^{\tau=s} h(s-\tau) g(\tau, u_{\underline{w}}(\tau))\, d\tau \Big] ds \Big](\theta) \Bigg\|_X d\theta$$

By using the conditions $(\mathbf{A_2})$, $(\mathbf{A_3} \cdot i)$, $(\mathbf{A_4})$, $(\mathbf{A_5} \cdot i)$ and $(\mathbf{A_8})$, one can get:

$$\left\|(F_{\underline{w}} u_{\underline{w}})(t) - u_0\right\|_X \le \rho' + M N_1 t + M h_\gamma N_2 t + M K_0 I_0 \int_{\theta=0}^{\theta=t} \Bigg\| \Big[v_0 - T(\gamma)u_0 - \int_{s=0}^{s=\gamma} T(\gamma-s)\Big[f(s, u_{\underline{w}}(s)) + \int_{\tau=0}^{\tau=s} h(s-\tau) g(\tau, u_{\underline{w}}(\tau))\, d\tau \Big] ds \Big](\theta) \Bigg\|_X d\theta$$

$$\|(F_{\underline{w}} u_{\underline{w}})(t) - u_0\|_X \le \rho' + M N_1 t + M h_\gamma N_2 t + M K_0 I_0 \int_{\theta=0}^{\theta=t} \Big[\|v_0\|_X - M\|u_0\|_X - M \int_{s=0}^{s=t} (N_1 + h_\gamma N_2)\, ds \Big] d\theta$$

$$\|(F_{\underline{w}} u_{\underline{w}})(t) - u_0\|_X \le \rho' + M N_1 \gamma + M h_\gamma N_2 \gamma + M K_0 I_0 \big[\|v_0\|_X - M\|u_0\|_X - M (N_1 + h_\gamma N_2)\ \gamma \big] \gamma \quad \ldots(15)$$

By using the condition $(\mathbf{A_9} \cdot i)$ and the equation defined in (15), one can get: $\|(F_{\underline{w}} u_{\underline{w}})(t) - u_0\|_X \le \rho$, for control function $\underline{w}(\cdot) \in L^2(J_0^*, O)$, $0 \le t \le \gamma$. Hence $F_{\underline{w}} u_{\underline{w}} \in S_{\underline{w}}$, i.e. $F_{\underline{w}} : S_{\underline{w}} \to S_{\underline{w}}$, for control function $\underline{w}(\cdot) \in L^2(J_0^*, O)$.

To complete the proof, the validation of step (3) is needed and as follow:

$$\|(F_{\underline{w}} \overline{u}_{\underline{w}})(t) - (F_{\underline{w}} \overline{\overline{u}}_{\underline{w}})(t)\|_X = \Big\| T(t)u_0 + \int_{s=0}^{s=t} T(t-s) \Big[f(s, \overline{u}_{\underline{w}}(s)) + \int_{\tau=0}^{\tau=s} h(s-\tau) g(\tau, \overline{u}_{\underline{w}}(\tau)) d\tau \Big] ds + \int_{\theta=0}^{\theta=t} T(t-\theta) B \tilde{G}^{-1} \Big[v_0 - T(\gamma)u_0 - \int_{s=0}^{s=t} T(\gamma - s) \Big[f(s, \overline{u}_{\underline{w}}(s)) + \int_{\tau=0}^{\tau=s} h(s-\tau) g(\tau, \overline{u}_{\underline{w}}(\tau))\, d\tau \Big] ds \Big](\theta) d\theta$$

$$- T(t)u_0 - \int_{s=0}^{s=t} T(t-s) \Big[f(s, \overline{\overline{u}}_{\underline{w}}(s)) + \int_{\tau=0}^{\tau=s} h(s-\tau) g(\tau, \overline{\overline{u}}_{\underline{w}}(\tau)) d\tau \Big] ds - \int_{\theta=0}^{\theta=t} T(t-\theta) B \tilde{G}^{-1} \Big[v_0 - T(\gamma)u_0 - \int_{s=0}^{s=t} T(\gamma - s) \Big[f(s, \overline{\overline{u}}_{\underline{w}}(s)) + \int_{\tau=0}^{\tau=s} h(s-\tau) g(\tau, \overline{\overline{u}}_{\underline{w}}(\tau))\, d\tau \Big] ds \Big](\theta)\, d\theta \Big\|_X$$

$$\|(F_{\underline{w}} \overline{u}_{\underline{w}})(t) - (F_{\underline{w}} \overline{\overline{u}}_{\underline{w}})(t)\|_X \le \int_{s=0}^{s=t} \Big\| T(t-s) \Big[[f(s, \overline{u}_{\underline{w}}(s)) - f(s, \overline{\overline{u}}_{\underline{w}}(s))] + \int_{\tau=0}^{\tau=s} h(s-\tau) [g(\tau, \overline{u}_{\underline{w}}(\tau)) - g(\tau, \overline{\overline{u}}_{\underline{w}}(\tau))] d\tau \Big] ds$$

$$+ \int_{\theta=0}^{\theta=t} \Big\| T(t-\theta) B \tilde{G}^{-1} \Big[-\int_{s=0}^{s=t} T(\gamma - s) \Big[[f(s, \overline{u}_{\underline{w}}(s)) - f(s, \overline{\overline{u}}_{\underline{w}}(s))] + \int_{\tau=0}^{\tau=s} h(s-\tau) [g(\tau, \overline{u}_{\underline{w}}(\tau)) - g(\tau, \overline{\overline{u}}_{\underline{w}}(\tau))] d\tau \Big] ds \Big](\theta) \Big\|_X d\theta$$

After simple calculations and using the condition $(\mathbf{A_3} \cdot i)$, $(\mathbf{A_4})$, $(\mathbf{A_5} \cdot i)$ and $(\mathbf{A_8})$ with equation defined in (8), one can get:

$$\left\|(F_{\underline{w}}\bar{u}_{\underline{w}})(t) - (F_{\underline{w}}\bar{\bar{u}}_{\underline{w}})(t)\right\|_X \le M\,L_0 \left\|\bar{u}_{\underline{w}}(s) - \bar{\bar{u}}_{\underline{w}}(s)\right\|_X \gamma + M\,h_\gamma\,L_1 \left\|\bar{u}_{\underline{w}}(\tau) - \bar{\bar{u}}_{\underline{w}}(\tau)\right\|_X \gamma + \left[M^2 K_0 I_0 \left[L_0 \left\|\bar{u}_{\underline{w}}(s) - \bar{\bar{u}}_{\underline{w}}(s)\right\|_X + h_\gamma L_1 \left\|\bar{u}_{\underline{w}}(\tau) - \bar{\bar{u}}_{\underline{w}}(\tau)\right\|_X \right]\gamma \right]\gamma$$

$$\left\|(F_{\underline{w}}\bar{u}_{\underline{w}})(t) - (F_{\underline{w}}\bar{\bar{u}}_{\underline{w}})(t)\right\|_X \le M\,L_0 \sup_{0\le t\le\gamma}\left\|\bar{u}_{\underline{w}}(t) - \bar{\bar{u}}_{\underline{w}}(t)\right\|_X \gamma + M\,h_\gamma\,L_1 \sup_{0\le t\le\gamma}\left\|\bar{u}_{\underline{w}}(t) - \bar{\bar{u}}_{\underline{w}}(t)\right\|_X \gamma + \left[M^2 K_0 I_0 \left[L_0 \sup_{0\le t\le\gamma}\left\|\bar{u}_{\underline{w}}(t) - \bar{\bar{u}}_{\underline{w}}(t)\right\|_X + h_\gamma L_1 \sup_{0\le t\le\gamma}\left\|\bar{u}_{\underline{w}}(t) - \bar{\bar{u}}_{\underline{w}}(t)\right\|_X \right]\gamma \right]\gamma$$

$$\left\|(F_{\underline{w}}\bar{u}_{\underline{w}})(t) - (F_{\underline{w}}\bar{\bar{u}}_{\underline{w}})(t)\right\|_X \le \left[M\,L_0\,\gamma + M\,h_\gamma\,L_1\,\gamma + M^2 K_0 I_0 \gamma^2 \left(L_0 + h_\gamma L_1\right) \right] \sup_{0\le t\le\gamma}\left\|\bar{u}_{\underline{w}}(t) - \bar{\bar{u}}_{\underline{w}}(t)\right\|_X$$

$$\left\|(F_{\underline{w}}\bar{u}_{\underline{w}})(t) - (F_{\underline{w}}\bar{\bar{u}}_{\underline{w}})(t)\right\|_X \le \left[M\,L_0\,\gamma^2 + M\,h_\gamma\,L_1\,\gamma^2 + M^2 K_0 I_0 \gamma^2 \left(L_0 + h_\gamma L_1\right) \right] \sup_{0\le t\le\gamma}\left\|\bar{u}_{\underline{w}}(t) - \bar{\bar{u}}_{\underline{w}}(t)\right\|_X$$

$$\left\|(F_{\underline{w}}\bar{u}_{\underline{w}})(t) - (F_{\underline{w}}\bar{\bar{u}}_{\underline{w}})(t)\right\|_X \le \left[M\,L_0 + M\,h_\gamma\,L_1 + M^2 K_0 I_0 \left(L_0 + h_\gamma L_1\right) \right] \gamma^2 \sup_{0\le t\le\gamma}\left\|\bar{u}_{\underline{w}}(t) - \bar{\bar{u}}_{\underline{w}}(t)\right\|_X$$

By using the condition $(\mathbf{A_9} \cdot \text{ii})$, one can get:

$$\left\|(F_{\underline{w}}\bar{u}_{\underline{w}})(t) - (F_{\underline{w}}\bar{\bar{u}}_{\underline{w}})(t)\right\|_X \le \left(1 - \frac{\rho'}{\rho}\right) \sup_{0\le t\le\gamma}\left\|\bar{u}_{\underline{w}}(t) - \bar{\bar{u}}_{\underline{w}}(t)\right\|_X \quad \dots (16)$$

Taking the sup-norm over $J_0^* = [0, \gamma]$ to the equation defined in (16), one can get:

$$\sup_{0\le t\le\gamma}\left\|(F_{\underline{w}}\bar{u}_{\underline{w}})(t) - (F_{\underline{w}}\bar{\bar{u}}_{\underline{w}})(t)\right\|_X \le \left(1 - \frac{\rho'}{\rho}\right) \sup_{0\le t\le\gamma}\left\|\bar{u}_{\underline{w}}(t) - \bar{\bar{u}}_{\underline{w}}(t)\right\|_X$$

, since $\|y\|_Y = \sup_{0\le t\le\gamma}\|y(t)\|_X$, we obtain:

$$\left\|F_{\underline{w}}\bar{u}_{\underline{w}} - F_{\underline{w}}\bar{\bar{u}}_{\underline{w}}\right\|_Y \le \left(1 - \frac{\rho'}{\rho}\right)\left\|\bar{u}_{\underline{w}} - \bar{\bar{u}}_{\underline{w}}\right\|_Y$$

,where $0 < \left(1 - \frac{\rho'}{\rho}\right) < 1$.

Thus $F_{\underline{w}}$ is a strict contraction map from $S_{\underline{w}}$ into $S_{\underline{w}}$ and therefore by the Banach fixed point theorem there is a unique $u_{\underline{w}} \in S_{\underline{w}}$ such that $F_{\underline{w}}u_{\underline{w}} = u_{\underline{w}}$,control function $\underline{w}(\cdot) \in L^2(J_0^*, O)$.

4. The Observability of Nonlinear Dynamic Control System

In the present work, the nonlinear control system has been concerned:

$$\frac{d\,u_w(t)}{dt} + A\,u_w(t) = f(t, u_w(t)) + \int_{s=0}^{s=t} h(t-s)\, g(s, u_w(s))\,ds + (B\,w)(t),\ t > 0$$

$$u_w(0) = u_0$$

$$y_w = Cu_w \quad \dots (17)$$

where A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\ge0}$ on a reflexive complex Banach space X into X, f and g are a nonlinear continuous maps defined from $[0,\gamma] \times X$ into X, h is the real valued continuous function defined from $[0,\gamma]$ into $\Re$ where $\Re$ is the set of all real numbers and the control function *w*(.) referred to as the input which is belong to $L^2(J_0^*, O)$, a Banach space of admissible control functions with O as a Banach space (control space) and $J_0^* = [0,\gamma]$, B is a bounded linear operator from O into X and the function $y_w(\cdot)$ is referred to as the output which is belong to Y , a reflexive complex Banach space of outputs , C is bounded linear operator defined from X into Y.

A continuous function $u_w(\cdot) \in C(J_0^* : X)$ which is defined in (6) is said to be a mild solution to the nonlinear control system defined in (17), so that the observability of the mild solution defined in (6) to the nonlinear control system defined in (17) have been established via Banach fixed point theorem and strongly continuous semigroup theory. The observability of the mild solution defined in (6) will be discussed and proved, i.e. the problem is whether it is possible to construct the state of the system $u_w(t), t \ge 0$, given the output $y_w(\cdot)$; for every control function *w*(.) belong to $L^2(J_0^*, O)$. Consider the homogenous linear part of (6), i.e. the system:

$$u(t) = T(t)u_0, t \in J_0^* = [0,\gamma] \quad \dots (18)$$

And

$$y(t) = CT(t)u_0, t \in J_0^* = [0,\gamma] \quad \dots (19)$$

Let us denotes $\Omega = C(J_0^* : Y)$ and define an operator H: X ⟶Y, by:

$$H u_0 = C T(t) u_0 \quad \dots (20)$$

Remarks 4.1 [1]

1. The system defined in (18) is said to be initially observable if kernel H = {0}.
2. The system defined in (18) is said to be continuously initially observable if $\|H u_0\|_Y = \|u_0\|_X$.
3. If a system defined in (18) is initially observable then the map H is injective but not surjective.
4. If the system defined in (18) is continuous initially observable then $H^{-1}: Y \longrightarrow X$ exists and bounded, i.e. there exists $R > 0$ such that $\|H^{-1} v\|_X \le R \|v\|_Y, \forall v \in Y$

Theorem 4.1 [13]

The system defined in (18) is continuously initially observable on $J_0^* = [0,\gamma]$ if and only if the control system define as follow:

$$u_w(t) = T(t) u_0 + \int_{s=0}^{s=t} T(t-s) B\, w(s)\, ds$$

is exactly controllable on $J_0^* = [0,\gamma]$.

Concluding Remark 4.1

1.It is clear that from theorem (3.2), the mild solution defined in (6) to the nonlinear control system defined in (17) is exactly controllability on $J_0^* = [0,\gamma]$, so as the special case, the nonlinear control system $u_w(t), t \ge 0$ given as follow:

$$u_w(t) = T(t) u_0 + \int_{s=0}^{s=t} T(t-s) B\, w(s)\, ds$$

is also exactly controllable on $J_0^* = [0,\gamma]$. Then by theorem (4.1), the system defined in (18) is continuously initially observable on $J_0^* = [0,\gamma]$, so by remarks 4.1.4 , the operator H: X ⟶Y, defined by: $H u_0 = C T(t) u_0$ be invertible and bounded .

2.Since the system defined in (18) is continuously initially observable, then the initial state u_0 of the system defined in (18) can be obtained as follow:

Taking the map H^{-1} from the left side to the system defined in (18), we get:

$$H^{-1}y(t) = H^{-1}CT(t)u_0, \quad t \in J_0^* = [0,\gamma]. \quad \ldots(21)$$

Since H=C T(t), so we get the initial state u_0 from the equation defined in (21):

$$u_0 = H^{-1}y(t), t \in J_0^* = [0,\gamma] \quad \ldots(22)$$

From the equation defined in (22) and the equation defined in (18), the nominal solution $u(t), t \geq 0$ become:

$$u(t) = T(t)H^{-1}y(t), \quad t \in J_0^* = [0,\gamma]$$

To generalize the result of the concluding Remark (4.1.2) to the nonhomogenous original nonlinear control system which is defined in (18), the following is adopted.

The problem formulation 4.1

The nonlinear observation $\tilde{y}_w(t) = C\,u_w(t)$ of the nonlinear control system defined in (6) can be expressed by substitutes (6) in $\tilde{y}_w(t)$, one can get:

$$\tilde{y}_w(t) = CT(t)u_0 + C\int_{s=0}^{s=t} T(t-s)\left[f(s, u_w(s)) + \int_{\tau=0}^{\tau=s} h(t-s)\, g(\tau, u_w(\tau))d\tau + B\,w(s) \right]ds \quad \ldots(23)$$

For all control function $w(.)$ belong to $L^2(J_0^*, O)$.

To obtain finite time observer, one can construct the initial state implicitly as a function of the state $u_w(\cdot)$ and found $u_w(\cdot)$ for arbitrary control function $w(.)$ belong to $L^2(J_0^*, O)$. The initial state u_0 of the system defined in (18) can be obtained as follow:

From the equation defined in (23), notice that:

$$CT(t)u_0 = \tilde{y}_w(t) - C\int_{s=0}^{s=t} T(t-s)\left[f(s, u_w(s)) + \int_{\tau=0}^{\tau=s} h(t-s)\, g(\tau, u_w(\tau))d\tau + B\,w(s) \right]ds \quad \ldots(24)$$

Since H is invertible operator, from the equation defined in (24), we have:

$$H^{-1}CT(t)u_0 = H^{-1}\left[\tilde{y}_w(t) - C\int_{s=0}^{s=t} T(t-s)\left[f(s, u_w(s)) + \int_{\tau=0}^{\tau=s} h(t-s)\, g(\tau, u_w(\tau))d\tau + B\,w(s) \right]ds \right] \quad \ldots(25)$$

From the equations defined in (20) and (25), one gets:

$$u_0 = H^{-1}\left[\tilde{y}_w(t) - C\int_{s=0}^{s=t} T(t-s)\left[f(s, u_w(s)) + \int_{\tau=0}^{\tau=s} h(t-s)\, g(\tau, u_w(\tau))d\tau + B\,w(s) \right]ds \right] \quad \ldots(26)$$

Substitute the equation defined in (26) in the equation defined in (6), we have that:

$$u_w(t) = T(t)H^{-1}\Big[\tilde{y}_w(t) - C \int_{s=0}^{s=t} T(t-s)\Big[f(s,u_w(s)) + \int_{\tau=0}^{\tau=s} h(t-s)\, g(\tau,u_w(\tau))d\tau + B\,w(s) \Big] ds \Big] + \int_{s=0}^{s=t} T(t-s)\Big[f(s,u_w(s)) + \int_{\tau=0}^{\tau=s} h(t-s)\, g(\tau,u_w(\tau))d\tau + B\,w(s) \Big] ds \quad \ldots(27)$$

Remark 4.2
The equation defined in (27) is the finite time observer as long as we can guarantee a fixed point for the mild solution $u_w(\cdot) \in C(J_0^* : X)$, $\forall$ control function $w(.) \in L^2(J_0^*, O)$.
The following main result is then developed.

Main Result 4.1
To study the observability of the mild solution defined in (6) to the nonlinear control system defined in (17), the previous results and the hypotheses $(\mathbf{A_1})$, $(\mathbf{A_3})$, $(\mathbf{A_4})$ and $(\mathbf{A_5})$ as well as the following hypotheses are adopted in the present work and as follow:

$(\mathbf{B_1})$ Let C is bounded linear operator defined from X into Y, i.e. there exists L>0 such that $\|Cx\|_Y \le L\|x\|_X$, $\forall x \in X$.

$(\mathbf{B_2})$ If $\gamma \in R^+$, {where R^+ the set of all positive real numbers}, satisfy the following conditions:

$(\mathbf{B_2} \cdot \mathrm{i})$

$$\gamma \le \frac{a - MRK_3}{M(1 - MRL)(N_1 + h_\gamma N_2 + K_0 K_1)}$$

, where a, M, R, K_3, L, N_1, N_2, K_0, h_γ and K_1 are positive real numbers such that $MRK_3 < a$ and $MRL \ne 1$

$(\mathbf{B_2} \cdot \mathrm{ii})$

$$\gamma \le \frac{\left(1 - \frac{a}{a+1}\right) - MRI_0}{M^2RL\,I_1 + M^2RL\,I_2\,h_\gamma + M\,I_1 + M\,h_\gamma\,I_2}$$

, where a, M, R, I_0, I_1, I_2, h_γ and L are positive real numbers such that $MRI_0 < \left(1 - \frac{a}{a+1}\right)$.

Lemma 4.1
Consider the nonlinear control system defined in (6). Let $Z = C(J_0^* : X)$, where Z is a complex Banach space with the sup–norm defined as follow:

$$\|z\|_Z = \sup_{0 \le t \le \gamma} \|z(t)\|_X$$

and assume M_w be a nonempty subset of Z which is depend on the control function $w(.) \in L^2(J_0^*, O)$, define as follow:

$$M_w = \{u_w \in Z : \|u_w(t)\|_X \le a \;; 0 \le t \le \gamma, a \in R^+\}$$

Then M_w is a closed subset of Z for every control function $w(.) \in L^2(J_0^*, O)$

Proof
Let the sequence $u_w^n \in M_w$ such that $u_w^n \xrightarrow{P.C} u_w$ as n→ ∞, where u_w^n is a sequence of continuous function depending on the control function $w(.) \in L^2(J_0^*, O)$ which is point wise convergent to u_w, to prove that $u_w \in M_w$, i.e. to prove the following steps:

Step (1) proves that $u_w \in Z$, $\forall$ control function $w(.) \in L^2(J_0^*, O)$.

Step (2) show that $\|u_w(t)\|_X \le a$, $\forall$ control function $w(.) \in L^2(J_0^*, O)$.

For the proof of step (1), since $u_w^n \in M_w$ depending on the continuity property with the point wise convergent of the sequence u_w^n, we have that the sequence u_w^n is uniformly convergent to u_w, depending on the uniformly convergent property, we have that $u_w \in Z$, $\forall$ $w(.) \in L^2(J_0^*, O)$, so step (1) holds.

To complete the proof, the validation of step (2) is needed and as follow, since the sequence u_w^n is uniform convergent to u_w and

$$\|u_w^n - u_w\|_Z = \sup_{0\le t\le\gamma}\|u_w^n(t) - u_w(t)\|_X ,$$

where Z is a complex Banach space then $\sup_{0\le t\le\gamma}\|u_w^n(t) - u_w(t)\|_X \to 0$ as $n\to\infty$, which implies that $\|u_w^n(t) - u_w(t)\|_X \to 0$ as $n\to\infty$, $\forall\, 0 \le t \le \gamma$ i.e. $\lim_{n\to\infty} u_w^n(t) = u_w(t)$, $\forall\, 0 \le t \le \gamma$.

For $0 \le t \le \gamma$, we have that:

$\|u_w(t)\|_X = \left\|\lim_{n\to\infty} u_w^n(t)\right\|_X = \lim_{n\to\infty}\| u_w^n(t)\|_X \le \lim_{n\to\infty} a = a$ .Hence $\|u_w(t)\|_X \le a$, $\forall$control function $w(.) \in L^2(J_0^*, O)$.We conclude that M_w is a closed subset of Z for every control function $w(.) \in L^2(J_0^*, O)$.

Lemma 4.2

Assume the hypotheses $(\mathbf{A_1})$, $(\mathbf{A_3})$, $(\mathbf{A_4})$, $(\mathbf{A_5})$ and $(\mathbf{B_1})$ holds. Consider the nonlinear observation $\tilde{y}_w(t)$ defined in (23) and $\overline{\tilde{y}}_w(t)$, $\overline{\overline{\tilde{y}}}_w(t)$ be the nonlinear observations defined as follow respectively:

$$\overline{\tilde{y}}_w(t) = CT(t)u_0 + C\int_{s=0}^{s=t} T(t-s)\left[f(s, \overline{u}_w(s)) + \int_{\tau=0}^{\tau=s} h(t-s)\, g(\tau, \overline{u}_w(\tau))d\tau + B\,w(s) \right]ds$$

$$\overline{\overline{\tilde{y}}}_w(t) = CT(t)u_0 + C\int_{s=0}^{s=t} T(t-s)\left[f(s, \overline{\overline{u}}_w(s)) + \int_{\tau=0}^{\tau=s} h(t-s)\, g(\tau, \overline{\overline{u}}_w(\tau))d\tau + B\,w(s) \right]ds$$

If $K_3 = LM\|u_0\|_X + LM(N_1 + h_\gamma N_2 + K_0 K_1)\gamma$ and $I_0 = (L M I_1 + L M I_2 h_\gamma)\gamma$.

Then

1. $\|\tilde{y}_w(t)\|_Y \le K_3$, $\forall$ control function $w(.) \in L^2(J_0^*, O)$.
2. $\left\|\overline{\tilde{y}}_w(t) - \overline{\overline{\tilde{y}}}_w(t)\right\|_Y \le I_0 \sup_{0\le t\le\gamma}\left\|\overline{u}_w(t) - \overline{\overline{u}}_w(t)\right\|_X$, for all control function $w(.) \in L^2(J_0^*, O)$, where K_3 and I_0 are positive real numbers.

Proof

For $0\le t\le\gamma$, notice that:

$$\|\tilde{y}_w(t)\|_Y = \Big\| CT(t)u_0 + C\int_{s=0}^{s=t} T(t-s)\Big[f(s,u_w(s)) + \int_{\tau=0}^{\tau=s} h(t-s)\, g(\tau,u_w(\tau))d\tau + B\,w(s)\Big]ds\Big\|_Y$$

$$\|\tilde{y}_w(t)\|_Y \le \|CT(t)u_0\|_Y + \Big\|C\int_{s=0}^{s=t} T(t-s)\Big[f(s,u_w(s)) + \int_{\tau=0}^{\tau=s} h(t-s)\, g(\tau,u_w(\tau))d\tau + B\,w(s)\Big]ds\Big\|_Y$$

By using the condition $(\mathbf{B_1})$, one gets:

$$\|\tilde{y}_w(t)\|_Y \le L\|T(t)u_0\|_Y + L\int_{s=0}^{s=t} \Big\|T(t-s)\Big[f(s,u_w(s)) + \int_{\tau=0}^{\tau=s} h(t-s)\, g(\tau,u_w(\tau))d\tau + B\,w(s)\Big]\Big\|_Y ds \quad \ldots(28)$$

From the equations defined in (8) and (28), one can get:

$$\|\tilde{y}_w(t)\|_Y \le L\,M\,\|u_0\|_X + L\,M\int_{s=0}^{s=t}\Big\| f(s,u_w(s)) + \int_{\tau=0}^{\tau=s} h(t-s)\, g(\tau,u_w(\tau))d\tau + B\,w(s)\Big\|_X ds$$

$$\|\tilde{y}_w(t)\|_Y \le L\,M\,\|u_0\|_X + L\,M\int_{s=0}^{s=t}\Big[\|f(s,u_w(s))\|_X + \int_{\tau=0}^{\tau=s} |h(t-s)|\, \|g(\tau,u_w(\tau))\|_X\, d\tau + \|B\,w(s)\|_X\Big]ds$$

By using the conditions $(\mathbf{A_3}\cdot \mathrm{i})$, $(\mathbf{A_4})$ and $(\mathbf{A_5})$, one can get:

$$\|\tilde{y}_w(t)\|_Y \le L\,M\,\|u_0\|_X + L\,M\int_{s=0}^{s=t}\Big[N_1 + h_\gamma N_2 + K_0 K_1\Big]ds$$

$$\|\tilde{y}_w(t)\|_Y \le L\,M\,\|u_0\|_X + L\,M\,[N_1 + h_\gamma N_2 + K_0 K_1]t,\ t \in [0,\gamma]$$

Hence

$$\|\tilde{y}_w(t)\|_Y \le L\,M\,\|u_0\|_X + L\,M\,[N_1 + h_\gamma N_2 + K_0 K_1]\gamma$$

From the assumption of the positive constant K_3, one can get: $\|\tilde{y}_w(t)\|_Y \le K_3$, $\forall$ control function $w(.)\in L^2(J_0^*, O)$. For the second part, one should notice that:

$$\left\| \overline{\tilde{y}}_w(t) - \overline{\overline{\tilde{y}}}_w(t) \right\|_Y = \Big\| \; CT(t)u_0 + C\int_{s=0}^{s=t} T(t-s)\Big[f\big(s,\overline{u}_w(s)\big) + \int_{\tau=0}^{\tau=s} h(t-s)\, g\big(\tau,\overline{u}_w(\tau)\big)d\tau + B\,w(s) \Big]ds - CT(t)u_0 - C\int_{s=0}^{s=t} T(t-s)\Big[f\big(s,\overline{\overline{u}}_w(s)\big) + \int_{\tau=0}^{\tau=s} h(t-s)\, g\big(\tau,\overline{\overline{u}}_w(\tau)\big)d\tau + B\,w(s) \Big]ds \Big\|_Y$$

$$\left\| \overline{\tilde{y}}_w(t) - \overline{\overline{\tilde{y}}}_w(t) \right\|_Y = \Big\| \; C\int_{s=0}^{s=t} T(t-s)\Big[f\big(s,\overline{u}_w(s)\big) - f\big(s,\overline{\overline{u}}_w(s)\big) \Big]ds + C\int_{s=0}^{s=t} T(t-s) \Big[\int_{\tau=0}^{\tau=s} h(t-s)\Big[g\big(\tau,\overline{u}_w(\tau)\big) - g\big(\tau,\overline{\overline{u}}_w(\tau)\big) \Big]d\tau \Big]ds \Big\|_Y$$

$$\left\| \overline{\tilde{y}}_w(t) - \overline{\overline{\tilde{y}}}_w(t) \right\|_Y \le \Big\| \; C\int_{s=0}^{s=t} T(t-s)\Big[f\big(s,\overline{u}_w(s)\big) - f\big(s,\overline{\overline{u}}_w(s)\big) \Big]ds \Big\|_Y + \Big\| C\int_{s=0}^{s=t} T(t-s) \Big[\int_{\tau=0}^{\tau=s} h(t-s)\Big[g\big(\tau,\overline{u}_w(\tau)\big) - g\big(\tau,\overline{\overline{u}}_w(\tau)\big) \Big]d\tau \Big]ds \Big\|_Y \quad \ldots(29)$$

From the equation defined in (29) and the condition $(\mathbf{B_1})$, one gets:

$$\left\| \overline{\tilde{y}}_w(t) - \overline{\overline{\tilde{y}}}_w(t) \right\|_Y \le L\int_{s=0}^{s=t} \Big\| \; T(t-s)\Big[f\big(s,\overline{u}_w(s)\big) - f\big(s,\overline{\overline{u}}_w(s)\big) \Big] \Big\|_X ds + L\int_{s=0}^{s=t} \Big\| T(t-s)\Big[\int_{\tau=0}^{\tau=s} h(t-s)\Big[g\big(\tau,\overline{u}_w(\tau)\big) - g\big(\tau,\overline{\overline{u}}_w(\tau)\big) \Big]d\tau \Big]ds \Big\|_X$$

From the equation defined in (8), we have:

$$\left\| \overline{\tilde{y}}_w(t) - \overline{\overline{\tilde{y}}}_w(t) \right\|_Y \le L\,M\int_{s=0}^{s=t} \Big\| \; f\big(s,\overline{u}_w(s)\big) - f\big(s,\overline{\overline{u}}_w(s)\big) \Big\|_X ds + L\,M \int_{s=0}^{s=t} \Big[\int_{\tau=0}^{\tau=s} |h(t-s)| \Big\| \; g\big(\tau,\overline{u}_w(\tau)\big) - g\big(\tau,\overline{\overline{u}}_w(\tau)\big) \Big\|_X d\tau \Big]ds \quad \ldots(30)$$

On using the conditions $(\mathbf{A_3} \cdot \text{ii})$, $(\mathbf{A_4})$ and the equation defined in (30), we get:

$$\left\| \tilde{\bar{y}}_w(t) - \tilde{\bar{\bar{y}}}_w(t) \right\|_Y \le L\,M\,I_1 \int_{s=0}^{s=t} \left\| \bar{u}_w(s) - \bar{\bar{u}}_w(s) \right\|_X ds + LM\, I_2\, h_\gamma \int_{s=0}^{s=t} \left\| \bar{u}_w(\tau) - \bar{\bar{u}}_w(\tau) \right\|_X ds$$

$$\left\| \tilde{\bar{y}}_w(t) - \tilde{\bar{\bar{y}}}_w(t) \right\|_Y \le L\,M\,I_1 \left\| \bar{u}_w(s) - \bar{\bar{u}}_w(s) \right\|_X t + LM\, I_2\, h_\gamma \left\| \bar{u}_w(\tau) - \bar{\bar{u}}_w(\tau) \right\|_X t$$

$, t \in J_0^* = [0, \gamma]$

Hence

$$\left\| \tilde{\bar{y}}_w(t) - \tilde{\bar{\bar{y}}}_w(t) \right\|_Y \le L\,M\,I_1 \left\| \bar{u}_w(s) - \bar{\bar{u}}_w(s) \right\|_X \gamma + LM\, I_2\, h_\gamma \left\| \bar{u}_w(\tau) - \bar{\bar{u}}_w(\tau) \right\|_X \gamma$$

$$\left\| \tilde{\bar{y}}_w(t) - \tilde{\bar{\bar{y}}}_w(t) \right\|_Y \le L\,M\,I_1\,\gamma \sup_{0\le t\le\gamma} \left\| \bar{u}_w(t) - \bar{\bar{u}}_w(t) \right\|_X + L\,M\,\gamma\, I_2\, h_\gamma \sup_{0\le t\le\gamma} \left\| \bar{u}_w(t) - \bar{\bar{u}}_w(t) \right\|_X$$

$$\left\| \tilde{\bar{y}}_w(t) - \tilde{\bar{\bar{y}}}_w(t) \right\|_Y \le \left(L\,M\,I_1 + L\,M\,I_2\,h_\gamma \right) \gamma \sup_{0\le t\le\gamma} \left\| \bar{u}_w(t) - \bar{\bar{u}}_w(t) \right\|_X$$

From the assumption of the positive constant I_0, one can get:

$$\left\| \tilde{\bar{y}}_w(t) - \tilde{\bar{\bar{y}}}_w(t) \right\|_Y \le I_0 \sup_{0\le t\le\gamma} \left\| \bar{u}_w(t) - \bar{\bar{u}}_w(t) \right\|_X$$

,∀ control function $w(.) \in L^2(J_0^*, O)$.

Theorem 4.1:

Assuming that the hypotheses $(\mathbf{A_1})$, $(\mathbf{A_3})$, $(\mathbf{A_4})$, $(\mathbf{A_5})$, $(\mathbf{B_1})$ and $(\mathbf{B_2})$ are hold. Then the nonlinear control system defined in (27) has a unique fixed point $u_w(\cdot) \in C(J_0^* : X)$, ∀ control function $w(.) \in L^2(J_0^*, O)$.

Proof

For arbitrary control function $w(.)$ belong to $L^2(J_0^*, O)$, define a nonlinear map $\Psi_w : M_w \to Z$ by:

$$(\Psi_w u_w)(t) = T(t) H^{-1}\left[\tilde{y}_w(t) - C\int_{s=0}^{s=t} T(t-s)\left[f(s, u_w(s)) + \int_{\tau=0}^{\tau=s} h(t-s)\, g(\tau, u_w(\tau))\,d\tau + B\,w(s) \right] ds \right] + \int_{s=0}^{s=t} T(t-s)\left[f(s, u_w(s)) + \int_{\tau=0}^{\tau=s} h(t-s)\, g(\tau, u_w(\tau))\,d\tau + B\,w(s) \right] ds \quad \ldots(31)$$

For all control function $w(.) \in L^2(J_0^*, O)$.

Our aim is then to prove that there exists a unique fixed point u_w of (31), i.e. there is a unique $u_w \in M_w$ such that $\Psi_w u_w = u_w$ for arbitrary control function $w(.) \in L^2(J_0^*, O)$.

The Banach fixed point theorem is adapted to ensure the existence of a unique fixed point u_w of (31), for arbitrary control function $w(.) \in L^2(J_0^*, O)$ and as following steps:

Step (1) M_w is closed subset of Z for arbitrary control function $w(.) \in L^2(J_0^*, O)$.

Step (2) $\Psi_w(M_w) \subseteq M_w$ for arbitrary control function $w(.) \in L^2(J_0^*, O)$.

Step (3) Ψ_w is a strict contraction on M_w for arbitrary control function $w(.) \in L^2(J_0^*, O)$. By using lemma (4.1), step (1) holds. To prove step (2), let u_w be the arbitrary element in M_w such that $\Psi_w u_w \in \Psi_w(M_w)$, to show that $\Psi_w u_w \in M_w$, the following are needed (see the definition of the set M_w in lemma 4.1).

1. $\Psi_w u_w \in Z, \forall w(.) \in L^2(J_0^*, O)$.
2. $\|\Psi_w u_w(t)\|_X \le a, \forall w(.) \in L^2(J_0^*, O)$, $0 \le t \le \gamma$. From the definition of the map Ψ_w which is defined in (31), it is clear that (1) holds. To prove (2), for $0 \le t \le \gamma$, notice that:

$$\|(\Psi_w u_w)(t)\|_X = \Bigg\| T(t) H^{-1}\Bigg[\tilde{y}_w(t) - C\int_{s=0}^{s=t} T(t-s)\Big[f(s, u_w(s)) + \int_{\tau=0}^{\tau=s} h(t-s)\, g(\tau, u_w(\tau))d\tau + B\,w(s) \Big]ds \Bigg] + \int_{s=0}^{s=t} T(t-s)\Big[f(s, u_w(s)) + \int_{\tau=0}^{\tau=s} h(t-s)\, g(\tau, u_w(\tau))d\tau + B\,w(s) \Big]ds \Bigg\|_X \quad ...(32)$$

From the equations defined in (8) and (32), we have:

$$\|(\Psi_w u_w)(t)\|_X \le M \Bigg\| H^{-1}\Bigg[\tilde{y}_w(t) - C\int_{s=0}^{s=t} T(t-s)\Big[f(s, u_w(s)) + \int_{\tau=0}^{\tau=s} h(t-s)\, g(\tau, u_w(\tau))d\tau + B\,w(s) \Big]ds \Bigg] \Bigg\|_X + M \int_{s=0}^{s=t} \Bigg\| f(s, u_w(s)) + \int_{\tau=0}^{\tau=s} h(t-s)\, g(\tau, u_w(\tau))d\tau + B\,w(s) \Bigg\|_X ds \quad ...(33)$$

From the equation defined in (33) and using the definition of bounded operator H^{-1}, one obtain the following:

$$\|(\Psi_w u_w)(t)\|_X \le M\,R \Bigg\| \tilde{y}_w(t) - C\int_{s=0}^{s=t} T(t-s)\Big[f(s, u_w(s)) + \int_{\tau=0}^{\tau=s} h(t-s)\, g(\tau, u_w(\tau))d\tau + B\,w(s) \Big]ds \Bigg\|_Y + M \int_{s=0}^{s=t} \Bigg\| f(s, u_w(s)) + \int_{\tau=0}^{\tau=s} h(t-s)\, g(\tau, u_w(\tau))d\tau + B\,w(s) \Bigg\|_X ds$$

A simple calculation and using the condition $(\mathbf{B_1})$, one can get:

$$\|(\Psi_w u_w)(t)\|_X \le M\ R\Big[\ \|\tilde{y}_w(t)\|_Y - L\int_{s=0}^{s=t}\Big\|\ T(t-s)\Big[\ f(s,u_w(s)) + \int_{\tau=0}^{\tau=s} h(t-s)\, g(\tau,u_w(\tau))d\tau + B\,w(s)\ \Big]ds\Big]\Big\|_Y + M\int_{s=0}^{s=t}\Big[\ \|f(s,u_w(s))\|_X + \int_{\tau=0}^{\tau=s}|h(t-s)|\,\|g(\tau,u_w(\tau))\|_X\, d\tau + \|B\,w(s)\|_X\ \Big]ds \quad \ldots(34)$$

From the equations defined in (8) and (34), one can get:

$$\|(\Psi_w u_w)(t)\|_X \le M\ R\Big[\ \|\tilde{y}_w(t)\|_Y - L\ M\int_{s=0}^{s=t}\Big[\ \|f(s,u_w(s))\|_X + \int_{\tau=0}^{\tau=s}|h(t-s)|\,\|g(\tau,u_w(\tau))\|_X\ d\tau + \|B\,w(s)\|_X\ \Big]ds\Big] + M\int_{s=0}^{s=t}\Big[\ \|f(s,u_w(s))\|_X + \int_{\tau=0}^{\tau=s}|h(t-s)|\,\|g(\tau,u_w(\tau))\|_X\ d\tau + \|B\,w(s)\|_X\ \Big]ds$$

By using the conditions $(\mathbf{A_3}\cdot i)$, $(\mathbf{A_4})$ and $(\mathbf{A_5})$, we get:

$$\|(\Psi_w u_w)(t)\|_X \le M\ R\Big[\ \|\tilde{y}_w(t)\|_Y - L\ M\left(N_1 + h_\gamma N_2 + K_0 K_1\right)t\ \Big] + M\left(N_1 + h_\gamma N_2 + K_0 K_1\right)t \quad \ldots(35)$$

By using lemma (4.2.1) and the equation defined in (35), notice that:

$$\|(\Psi_w u_w)(t)\|_X \le M\ R\ K_3 + M\left(1 - M\ R\ L\right)\left(N_1 + h_\gamma N_2 + K_0 K_1\right)t$$

$, t\in J_0^* = [0,\gamma]$. Hence

$$\|(\Psi_w u_w)(t)\|_X \le M\ R\ K_3 + M\left(1 - M\ R\ L\right)\left(N_1 + h_\gamma N_2 + K_0 K_1\right)\gamma \quad \ldots(36)$$

By using the condition $(\mathbf{B_2}\cdot i)$ and the equation defined in (36), one can get: $\|\Psi_w u_w(t)\|_X \le a$, $\forall\ w(.)\in L^2(J_0^*, O)$, $\forall 0\le t\le\gamma$.

Hence $\Psi_w : M_w \to M_w$, $\forall$ control function $w(.)\in L^2(J_0^*, O)$.

To prove step (3), let $\bar{u}_w, \bar{\bar{u}}_w \in M_w$, where $\bar{u}_w, \bar{\bar{u}}_w$ are the continuous function depend on the control function $w(.)\in L^2(J_0^*, O)$, then:

$$\left\|(\Psi_w \bar{u}_w)(t) - (\Psi_w \bar{\bar{u}}_w)(t)\right\|_X =$$

$$\Big\| \; T(t)\,H^{-1}\Big[\; \tilde{\bar{y}}_w(t) - C \int_{s=0}^{s=t} T(t-s)\Big[\; f\big(s,\bar{u}_w(s)\big) + \int_{\tau=0}^{\tau=s} h(t-s)\,g\big(\tau,\bar{u}_w(\tau)\big)d\tau + B\,w(s) \;\Big]ds\Big]$$

$$+ \int_{s=0}^{s=t} T(t-s)\Big[\; f\big(s,\bar{u}_w(s)\big) + \int_{\tau=0}^{\tau=s} h(t-s)\,g\big(\tau,\bar{u}_w(\tau)\big)d\tau + B\,w(s) \;\Big]ds - T(t)\,H^{-1}$$

$$\Big[\; \tilde{\bar{\bar{y}}}_w(t) - C \int_{s=0}^{s=t} T(t-s)\Big[f\big(s,\bar{\bar{u}}_w(s)\big) + \int_{\tau=0}^{\tau=s} h(t-s)\, g\big(\tau,\bar{\bar{u}}_w(\tau)\big)d\tau + B\,w(s) \;\Big]ds\Big]$$

$$- \int_{s=0}^{s=t} T(t-s)\Big[\; f\big(s,\bar{\bar{u}}_w(s)\big) + \int_{\tau=0}^{\tau=s} h(t-s)\,g\big(\tau,\bar{\bar{u}}_w(\tau)\big)d\tau + B\,w(s) \;\Big]ds \Big\|_X$$

$$\left\|(\Psi_w \bar{u}_w)(t) - (\Psi_w \bar{\bar{u}}_w)(t)\right\|_X =$$

$$\Big\| \; T(t)\,H^{-1}\Big[\; \tilde{\bar{y}}_w(t) - \tilde{\bar{\bar{y}}}_w(t) \;\Big]$$

$$+ T(t)\,H^{-1}\Big[\; -C\int_{s=0}^{s=t} T(t-s)\Big[f\big(s,\bar{u}_w(s)\big) - f\big(s,\bar{\bar{u}}_w(s)\big) \;\Big]ds \;\Big]$$

$$+ T(t)\,H^{-1}\Big[\; -C\int_{s=0}^{s=t} T(t-s)\Big[\int_{\tau=0}^{\tau=s} h(t-s)\Big[\; g\big(\tau,\bar{u}_w(\tau)\big) - g\big(\tau,\bar{\bar{u}}_w(\tau)\big) \;\Big]d\tau \;\Big]ds \;\Big]$$

$$+\int_{s=0}^{s=t} T(t-s)\Big[\; f\left(s,\overline{u}_w(s)\right)$$

$$-f\left(s,\overline{u}_w(s)\right)\Big]ds+\int_{s=0}^{s=t} T(t-s)$$

$$\Big[\int_{\tau=0}^{\tau=s} h(t-s)\Big[\; g\left(\tau,\overline{u}_w(\tau)\right)-$$

$$g\left(\tau,\overline{\overline{u}}_w(\tau)\right)\Big\|d\tau\;\Big]ds\;\Big\|_X$$

$$\left\|(\Psi_w\overline{u}_w)(t)-(\Psi_w\overline{\overline{u}}_w)(t)\right\|_X \le$$

$$\Big\|\; T(t)H^{-1}\Big[\;\tilde{\overline{y}}_w(t)-\tilde{\overline{\overline{y}}}_w(t)\;\Big]\Big\|$$

$$+\Big\|T(t)H^{-1}\Big[\;-C\int_{s=0}^{s=t} T(t-s)\Big[$$

$$f\left(s,\overline{u}_w(s)\right)-f\left(s,\overline{\overline{u}}_w(s)\right)\Big]ds\;\Big]\Big\|$$

$$+\Big\|T(t)H^{-1}\Big[\;-C\int_{s=0}^{s=t} T(t-s)\Big[$$

$$\int_{\tau=0}^{\tau=s} h(t-s)\Big[\; g\left(\tau,\overline{u}_w(\tau)\right)-$$

$$g\left(\tau,\overline{\overline{u}}_w(\tau)\right)\Big]d\tau\;\Big]ds\;\Big]\Big\|_X$$

$$+\int_{s=0}^{s=t}\Big\|\; T(t-s)\Big[\; f\left(s,\overline{u}_w(s)\right)$$

$$-f\left(s,\overline{u}_w(s)\right)\Big]\;\Big\|ds+\int_{s=0}^{s=t}\Big\|\; T(t-s)$$

$$\Big[\int_{\tau=0}^{\tau=s} h(t-s)\Big[\; g\left(\tau,\overline{u}_w(\tau)\right)-$$

$$g\left(\tau,\overline{\overline{u}}_w(\tau)\right)\Big\|d\tau\;\Big]\;\Big\|_X ds$$

From the equation defined in (8) and using the definition of bounded operator H^{-1}, we have:

$$\left\|(\Psi_w\overline{u}_w)(t)-(\Psi_w\overline{\overline{u}}_w)(t)\right\|_X \le$$

$$M\,R\left\|\tilde{\overline{y}}_w(t)-\tilde{\overline{\overline{y}}}_w(t)\right\|_X+M\,R$$

$$\Big\|-C\int_{s=0}^{s=t} T(t-s)\Big[\; f\left(s,\overline{u}_w(s)\right)-$$

$$f\left(s,\overline{\overline{u}}_w(s)\right)\Big]ds\;\Big\|_X+M\,R$$

$$\Big\|-C\int_{s=0}^{s=t} T(t-s)\Big[\int_{\tau=0}^{\tau=s} h(t-s)$$

$$\Big[\; g\left(\tau,\overline{u}_w(\tau)\right)-g\left(\tau,\overline{\overline{u}}_w(\tau)\right)\Big]$$

$$d\tau\;\Big]ds\;\Big]\Big\|_X+$$

$$M\int_{s=0}^{s=t}\Big\|\; f\left(s,\overline{u}_w(s)\right)-$$

$$-f\left(s,\overline{u}_w(s)\right)\Big\|ds+M\int_{s=0}^{s=t}\Big\|$$

$$\Big\|\int_{\tau=0}^{\tau=s} h(t-s)\Big[\; g\left(\tau,\overline{u}_w(\tau)\right)-$$

$$g\left(\tau,\overline{\overline{u}}_w(\tau)\right)\Big]d\tau\;\Big\|_X ds$$

By using the condition $(\mathbf{B_1})$ and some simplifications, we obtain:

$$\left\|(\Psi_w\overline{u}_w)(t)-(\Psi_w\overline{\overline{u}}_w)(t)\right\|_X \le$$

$$M\,R\left\|\tilde{\overline{y}}_w(t)-\tilde{\overline{\overline{y}}}_w(t)\right\|_X+M\,R\,L$$

$$\int_{s=0}^{s=t}\Big\|T(t-s)\Big[\; f\left(s,\overline{u}_w(s)\right)-$$

$$f\left(s, \overline{\overline{u}}_w(s)\right) \Big] \Big\|_X ds + M R L \int_{s=0}^{s=t} \Big\| T(t-s) \Big[\int_{\tau=0}^{\tau=s} h(t-s) \Big[g\left(\tau, \overline{u}_w(\tau)\right) - g\left(\tau, \overline{\overline{u}}_w(\tau)\right) \Big] d\tau \Big] \Big]\Big\|_X ds +$$

$$M \int_{s=0}^{s=t} \Big\| f\left(s, \overline{u}_w(s)\right) - f\left(s, \overline{\overline{u}}_w(s)\right) \Big\| ds + M \int_{s=0}^{s=t} \Big[\int_{\tau=0}^{\tau=s} |h(t-s)| \Big\| g\left(\tau, \overline{u}_w(\tau)\right) - g\left(\tau, \overline{\overline{u}}_w(\tau)\right) \Big\|_X d\tau \Big] ds$$

Using the equation defined in (8), we get:

$$\left\|(\Psi_w \overline{u}_w)(t) - (\Psi_w \overline{\overline{u}}_w)(t)\right\|_X \le M R \left\|\overline{\tilde{y}}_w(t) - \overline{\overline{\tilde{y}}}_w(t)\right\|_X + M^2 R L \int_{s=0}^{s=t} \left\| f\left(s, \overline{u}_w(s)\right) - f\left(s, \overline{\overline{u}}_w(s)\right)\right\|_X ds$$

$$+ M^2 R L \int_{s=0}^{s=t} \Big[\int_{\tau=0}^{\tau=s} |h(t-s)| \left\|g\left(\tau, \overline{u}_w(\tau)\right) - g\left(\tau, \overline{\overline{u}}_w(\tau)\right)\right\|_X d\tau \Big] ds$$

$$+ M \int_{s=0}^{s=t} \Big\| f\left(s, \overline{u}_w(s)\right) - f\left(s, \overline{\overline{u}}_w(s)\right) \Big\| ds + M \int_{s=0}^{s=t} \Big[\int_{\tau=0}^{\tau=s} |h(t-s)| \Big\| g\left(\tau, \overline{u}_w(\tau)\right) - g\left(\tau, \overline{\overline{u}}_w(\tau)\right) \Big\|_X d\tau \Big] ds$$

From the lemma (4.2.2) and using the conditions $(\mathbf{A_3} \cdot \text{ii})$ and $(\mathbf{A_4})$, one can get:

$$\left\|(\Psi_w \overline{u}_w)(t) - (\Psi_w \overline{\overline{u}}_w)(t)\right\|_X \le M R I_0 \sup_{0\le t\le\gamma} \left\|\overline{u}_w(t) - \overline{\overline{u}}_w(t)\right\|_X + M^2 R L I_1 \int_{s=0}^{s=t} \left\|\overline{u}_w(t) - \overline{\overline{u}}_w(t)\right\|_X ds$$

$$+ M^2 R L I_2 h_\gamma \int_{s=0}^{s=t} \left\|\overline{u}_w(t) - \overline{\overline{u}}_w(t)\right\|_X ds + M I_1 \int_{s=0}^{s=t} \left\|\overline{u}_w(t) - \overline{\overline{u}}_w(t)\right\|_X ds + M h_\gamma I_2 \int_{s=0}^{s=t} \left\|\overline{u}_w(t) - \overline{\overline{u}}_w(t)\right\|_X ds$$

$$\left\|(\Psi_w \overline{u}_w)(t) - (\Psi_w \overline{\overline{u}}_w)(t)\right\|_X \le M R I_0 \sup_{0\le t\le\gamma} \left\|\overline{u}_w(t) - \overline{\overline{u}}_w(t)\right\|_X + M^2 R L I_1 \sup_{0\le t\le\gamma} \left\|\overline{u}_w(t) - \overline{\overline{u}}_w(t)\right\|_X t + M^2 R L I_2 h_\gamma \sup_{0\le t\le\gamma} \left\|\overline{u}_w(t) - \overline{\overline{u}}_w(t)\right\|_X t$$

$$+ M I_1 \sup_{0\le t\le\gamma} \left\|\overline{u}_w(t) - \overline{\overline{u}}_w(t)\right\|_X t + M h_\gamma I_2 \sup_{0\le t\le\gamma} \left\|\overline{u}_w(t) - \overline{\overline{u}}_w(t)\right\|_X t$$

$$\left\|(\Psi_w\overline{u}_w)(t)-(\Psi_w\overline{\overline{u}}_w)(t)\right\|_X \le$$
$$M\,R\,I_0 \sup_{0\le t\le\gamma}\left\|\overline{u}_w(t)-\overline{\overline{u}}_w(t)\right\|_X + M^2 R\,L\,I_1 \sup_{0\le t\le\gamma}\left\|\overline{u}_w(t)-\overline{\overline{u}}_w(t)\right\|_X\gamma + M^2 R\,L\,I_2\,h_\gamma \sup_{0\le t\le\gamma}\left\|\overline{u}_w(t)-\overline{\overline{u}}_w(t)\right\|_X\gamma$$
$$+ M\,I_1 \sup_{0\le t\le\gamma}\left\|\overline{u}_w(t)-\overline{\overline{u}}_w(t)\right\|_X\gamma + M\,h_\gamma\,I_2 \sup_{0\le t\le\gamma}\left\|\overline{u}_w(t)-\overline{\overline{u}}_w(t)\right\|_X\gamma$$

$$\left\|(\Psi_w\overline{u}_w)(t)-(\Psi_w\overline{\overline{u}}_w)(t)\right\|_X \le \{M\,R\,I_0 + M^2 R\,L\,I_1\,\gamma + M^2 R\,L\,I_2\,h_\gamma\,\gamma + M\,I_1\,\gamma + M\,h_\gamma\,I_2\,\gamma\} \sup_{0\le t\le\gamma}\left\|\overline{u}_w(t)-\overline{\overline{u}}_w(t)\right\|_X$$

$$\left\|(\Psi_w\overline{u}_w)(t)-(\Psi_w\overline{\overline{u}}_w)(t)\right\|_X \le \left\{ M\,R\,I_0 + \left(M^2 R\,L\,I_1 + M^2 R\,L\,I_2 h_\gamma + M\,I_1 + M\,h_\gamma\,I_2\right)\gamma \right\} \sup_{0\le t\le\gamma}\left\|\overline{u}_w(t)-\overline{\overline{u}}_w(t)\right\|_X$$

By using the condition $(\mathbf{B}_2 \cdot \text{ii})$, we get:

$$\left\|(\Psi_w\overline{u}_w)(t)-(\Psi_w\overline{\overline{u}}_w)(t)\right\|_X \le \left(1-\frac{a}{1+a}\right)\sup_{0\le t\le\gamma}\left\|\overline{u}_w(t)-\overline{\overline{u}}_w(t)\right\|_X \quad \ldots (37)$$

Taking the sup-norm over [0,γ] to the equation defined in (37), we have:

$$\sup_{0\le t\le\gamma}\left\|(\Psi_w\overline{u}_w)(t)-(\Psi_w\overline{\overline{u}}_w)(t)\right\|_X \le \left(1-\frac{a}{1+a}\right)\sup_{0\le t\le\gamma}\left\|\overline{u}_w(t)-\overline{\overline{u}}_w(t)\right\|_X$$

, since $\|z\|_Z = \sup_{0\le t\le\gamma}\|z(t)\|_X$, we obtain:

$$\left\|\Psi_w\overline{u}_w - \Psi_w\overline{\overline{u}}_w\right\|_Z \le \left(1-\frac{a}{1+a}\right)\left\|\overline{u}_w - \overline{\overline{u}}_w\right\|_Z$$

, where $0 < \left(1-\frac{a}{1+a}\right) < 1$.

Thus Ψ_w is a strict contraction map from M_w into M_w and therefore by the Banach fixed point theorem there is a unique $u_w \in M_w$ such that $\Psi_w u_w = u_w$ for arbitrary control function $w(.) \in L^2(J_0^*, O)$.

Concluding Remark 4.2

To prove that the nonlinear control system defined in (6) is observable on $J_0^* = [0,\gamma]$, recall the equation defined in (31), as follow:

$$(\Psi_w u_w)(t) = T(t)H^{-1}\Big[\ \tilde{y}_w(t) - C\int_{s=0}^{s=t} T(t-s)\Big[\ f(s,u_w(s)) + \int_{\tau=0}^{\tau=s} h(t-s)\,g(\tau,u_w(\tau))d\tau + B\,w(s)\ \Big]ds\Big] + \int_{s=0}^{s=t} T(t-s)\Big[\ f(s,u_w(s)) + \int_{\tau=0}^{\tau=s} h(t-s)\,g(\tau,u_w(\tau))d\tau + B\,w(s)\ \Big]ds$$

Multiply the operator C from the left side to the equation defined in (31), we get:

$$C(\Psi_w u_w)(t) = CT(t)H^{-1}\Big[\tilde{y}_w(t) - C\int_{s=0}^{s=t} T(t-s)\Big[f(s,u_w(s)) + \int_{\tau=0}^{\tau=s} h(t-s)g(\tau,u_w(\tau))d\tau + Bw(s)\Big]ds\Big] + C\int_{s=0}^{s=t} T(t-s)\Big[f(s,u_w(s)) + \int_{\tau=0}^{\tau=s} h(t-s)g(\tau,u_w(\tau))d\tau + Bw(s)\Big]ds \qquad \ldots(38)$$

From the equation defined in (20), we have $H = CT(t)$, hence the equation defined in (38) become $C(\Psi_w u_w)(t) = \tilde{y}_w(t)$, $\forall$control function $w(.)\in L^2(J_0^*,O)$.
From the theorem (4.1), the nonlinear map Ψ_w has a unique fixed point, i.e. $\Psi_w u_w = u_w$, then we have: $C\, u_w(t) = \tilde{y}_w(t)$,$\forall$control function $w(.)\in L^2(J_0^*,O)$. So we conclude that the nonlinear control system defined in (6) is observable on $J_0^* = [0,\gamma]$.

5. Conclusions:

1. The basic preliminaries of understanding this subject are infinite dimensional spaces and theory of semigroup and some non-linear functional analysis. This subject is very important in applications in control theory area.
2. The existence and uniqueness, controllability as well as the observability problems have represented the main objects of real life dynamical control system and its applications to applied mathematics, physics and engineering.
3. This approach is limited to some class of a nonlinear dynamical control system, where the nonlinearity should satisfy some required condition, the nominal linear part defined in (18) must satisfy the observability condition in infinite dimensional space.
4. This approach and its theoretical proof have depended completely on the theory of strongly continuous semigroup and Banach fixed point theorem. This result may be generalized for important classes of the semigroup theory, analytic semigroup and compact semigroup.
5. The relation between the exactly controllable and continuously initially observable have founded the results of the nominal system defined in (18) and its nonlinearity control system defined in (6), which leads to the concept of nonlinear observability which is defined in (23).

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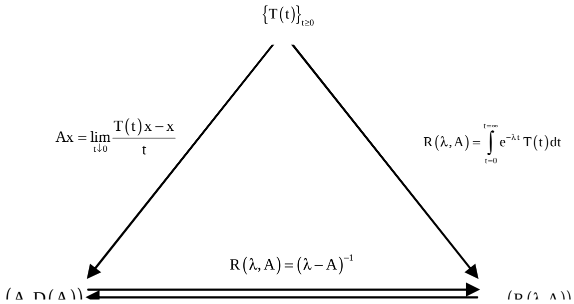

The diagram (1) represents the relations among a semigroup, its generator and its resolvent.