Shifted Chybeshev Polynomials for a Certain System of Fractional Order Integro-Differential Equations

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Abstract

The main goal of this paper lies briefly in submitting and modifying some numerical methods for solving system of linear Fractional order Integro- Differential Equations of Fredholm type (L.FFIDE's). in this method four kinds of shifted Chybeshev polynomials (T^*,U^*,$V^*$and W^***)** are used as a bases of independed polynomials approximation $f_n(x)$ .The general fractional derivatives of these polynomials are formulated $(D^a T_n^*, D^a U_n^*, D^a V_n^* andD^a W_n^*)$ in the framework of the Riemann-liouville definition .Some numerical examples are solving to show that the different between these polynomials , furthermore Algorithms and programs by using MATLAB program are given .

Keywords : Shifted Chybeshev Polynomials, System of fractional integro-differential equations**.**

دوال شيبيشيف المعدلة لحل نظام محدد لمعادلات تكاملية تفاضلية كسورية

الخلاصة

الهدف الاساسي من هذا البحث هو تقديم وتطوير بعض الطرق العددية لحل نظام المعادلات التفاضلية التكاملية الكسورية الخطية من النوع فريدهولم. استخدمت اربعة انواع من الدوال المتعامدة شيبيشيف كدوال لتقريب الدوال المجهولة . تم ايجاد المشتقات الكسورية لهذه الدوال الاربعة باستخدام تعريف ريمان . تم اعطاء بعض الامثلة التوضيحية لبيان الفروقات في النتائج بين دوال التقريب . علاوة على ذلك تم اعطاء خطوات الحل المتبعة في ايجاد الحل العددي باستخدام برنامج الماتلاب.

1-Introduction

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order (real and complex numbers). The term fractional is a misnomer, but it is retained following the prevailing use.

System of fractional integro-differential equations are equations having unknown function together with both fractional differential and integral operations and has the form:

$$D_x^a f^i(x) = g^i(x) + \sum_{j=1}^{m} \int_a^b k_{ij}(x,t) f^j(t)dt$$

i=1,2,…,m ; $a \le x \le b$

The theory and application of fractional integral and derivatives can be fund in many fields of science and engineering, such as Viscoelasticity, fractional differential which have been used to describe material's

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constitutive equations. In fact a well-known equation which contains a fractional integral operator is the Abel integral equation.[1,2,4,5,9].

2-Basic Definitions: In this section we give definitions:

2-1 Definition: the fractional derivative by Reimaan-Lovill (R-L) has the form:

$$D_x^a F(x)=\begin{cases}\dfrac{d^m}{dt^m}\left[\dfrac{1}{\Gamma(m-a)}\displaystyle\int_0^t (t-x)^{m-a-1}F(x)dx\right] & m-1<a<m\\[2mm] \dfrac{d^m}{dt^m}F(x) & m=a\end{cases}$$

Where m is an integer number less than α. [**3, 5, 7**]

Note: In this paper, we use ***R-L*** definition and its properties to find fractional derivatives of polynomials.

2-2 Some important properties of operator (D_x^a) [7,8]:

a- $$D_x^a \sum c_i f_i(x)=\sum_{i=1}^{n} c_i D_x^a f_i(x)$$

the linearity property.

b- $$D_x^a c=\frac{c}{\Gamma(n-a)}\frac{d^n}{dx^n}\left(\frac{x^{n-a}}{n-a}\right)$$

where c is constant.

c- $$D_x^a x^m=\frac{m!}{\Gamma(m-a+1)}x^{m-a}$$

where m=0, 1, 2,…, $\alpha > 0$.

In a special case, $\alpha = 0.5$ we have:

d- $$D_x^{0.5} c=\frac{c}{\sqrt{p\,x}}$$

e- $$D_x^{0.5} x^m=\frac{(m!)^2(4x)^m}{(2m)!\sqrt{p\,x}}$$

m=0, 1, 2,…

3- Chebyshev polynomials [3]: Chebyshev polynomials are orthogonal functions, and every where dense in numerical analysis. These polynomials have four kinds and have the forms:

3-1 First kind $T_n(x)$ is a polynomial in x of degree n. Defined by the relation:

$T_n(x) = \cos(n\theta)$ where $x = \cos\theta$ …(1)

The range of the variable x is the interval [-1,1], and the range of θ can be taken as $[0,\pi]$.

The recurrence relation:

$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$; $n=2, 3,\dots$ where $T_0(x)=1$ and $T_1(x)=x$ …(2)

The general form given by:

$$T_n(x)=\frac{n}{2}\sum_{r=0}^{[n/2]}(-1)^r\frac{(n-r-1)!}{r!(n-2r)!}(2x)^{(n-2r)}$$

; $T_0(x)=1$; $n > 1$ …(3)

3-2 Second kind polynomial $U_n(x)$: It is a polynomial of degree n in x defined by:

$U_n(x) = \sin(n+1)\theta/\sin\theta$ where $x = \cos\theta$ … (4)

The recurrence relation:

$U_n(x)=2xU_{n-1}(x)- U_{n-2}(x)$; $n=2,3,\dots$ where $U_0(x)=1$; $U_1(x)=2x$ …(5)

The general form:

$$U_n(x)=\sum_{r=0}^{[n/2]}(-1)^r\frac{(n-r)!}{r!(n-2r)!}(2x)^{(n-2r)}$$;

$U_0(x)=1$; $n > 1$ …(6)

3-3 Third kind polynomial $V_n(x)$: Are polynomials of degree n in x defined by:

$V_n(x)=\cos(n+1/2)\theta / \cos(1/2\,\theta)$

Where $x= \cos\theta$ …(7)

The recurrence relation:

$V_n(x)=2xV_{n-1}(x)-V_{n-2}(x)$; $n=2,3,\dots$ where $V_0(x)=1$; $V_1(x)=2x-1$ …(8)

3-4 Fourth **kind polynomial** $W_n(x)$: are polynomials of degree n in x defined by :

$W_n(x) = \sin(n+1/2)\theta/\sin(1/2\,\theta)$ where $x=\cos\theta$ … (9)

The recurrence relation:

$W_n(x) = 2xW_{n-1}(x) - W_{n-2}(x)$; $n=2,3,..$ Where $W_0(x)=1$; $W_1(x)=2x+1$ …(10)

4-Connections between the four kinds of polynomials:

Since $\sin(n+1)\theta - \sin(n-1)\theta = 2\sin\theta \cos n\theta$ we have :

$U_n(x) - U_{n-2}(x) = 2T_n(x)$; $n=2,3,\ldots$ where $U_0(x) = T_0(x) = 1$ …(11)

$U_n(x) = 1/2\ [\ V_n(x) + W_n(x)\ ]$ $n=2,3,\ldots$ …(12)

Since the trigonometric relations

$2\sin(1/2\ \theta)\cos(n+1/2)\theta = \sin(n+1)\theta - \sin(n\theta)$

$2\cos(1/2\ \theta)\sin(n+1/2)\theta = \sin(n+1)\theta + \sin(n\theta)$

We have

$V_n(x) = U_n(x) - U_{n-1}(x)$ where $U_0(x)=1$; $U_1(x)=2x$ …(13)

$W_n(x) = U_n(x) + U_{n-1}(x)$ where $U_0(x)=1$; $U_1(x)=2x$ …(14)

5-The shifted Chebyshev polynomials (T_n^*, U_n^*, V_n^* and W_n^*) [10] **:**

Since the range [0,1] is quite often more convenient to use than range [-1,1] , we sometimes map the independed variable ***x*** in [0,1] to the variable ***s*** in [-1,1] by the transformations

$S = 2x-1$ or $x = 1/2\ (1+s)$ … (15)

Using the variable s into equations above (3,6,13 and 14) we have four forms of shifted Chebyshev polynomials,eq's (16-19).

$\mathbf{T_n^*(x) = T_n(s)}$; $\mathbf{U_n^*(x) = U_n(s)}$; $\mathbf{V_n^*(x) = V_n(s)}$; $\mathbf{W_n^*(x) = W_n(s)}$

where s=2x-1

And the general forms:

$$\mathbf{T_n^*(x)} = \frac{n}{2}\sum_{r=0}^{[n/2]}(-1)^r\frac{(n-r-1)!}{r!(n-2r)!}(2s)^{(n-2r)} \quad ;$$

$T_0(s)=1$; $n > 1$ …(16)

$$\mathbf{U_n^*}(x) = \sum_{r=0}^{[n/2]}(-1)^r\frac{(n-r)!}{r!(n-2r)!}(2s)^{(n-2r)} \quad ;$$

$U_0(s)=1$; $n > 1$ …(17)

$\mathbf{V_n^*}(x) = U_n(s) - U_{n-1}(s)$ where $U_0(s)=1$; $U_1(s)=2s$ …(18)

$\mathbf{W_n^*}(x) = U_n(s) + U_{n-1}(s)$ where $U_0(s)=1$; $U_1(s)=2s$ …(19)

6- Fractional Derivatives of shifted Chebyshev polynomials:

Consider the properties of the operator ($\mathbf{D}_x^a$) eq's(1-2 a, b, and c) putting in eq's (16,17,18 and 19) yield:

$$\mathbf{D}_x^a\mathbf{T}_n^*\mathbf{(x)} = \frac{n}{2}\sum_{r=0}^{[n/2]}(-1)^r\frac{\Gamma(n-r)(2)^{(n-2r)}}{r!\Gamma(n-2r+a)}(s)^{(n-2r-a)}$$

; $n > 1$ …(20)

where $\mathbf{D}_x^a\mathbf{T}_0^*\mathbf{(x)} = \dfrac{1}{\Gamma(n-a)}\dfrac{d^n}{dx^n}\left(\dfrac{s^{n-a}}{n-a}\right)$; where $s=2x-1$ …(20a)

$$\mathbf{D}_x^a\mathbf{U}_n^*\mathbf{(x)} = \sum_{r=0}^{[n/2]}(-1)^r\frac{\Gamma(n-r)(2)^{(n-2r)}}{r!\Gamma(n-2r+a)}(s)^{(n-2r-a)}$$

; $n > 1$ …(21)

where $\mathbf{D}_x^a\mathbf{U}_0^*\mathbf{(x)} = \dfrac{1}{\Gamma(n-a)}\dfrac{d^n}{dx^n}\left(\dfrac{s^{n-a}}{n-a}\right)$; for s=2x- …(21a)

$\mathbf{D}_x^a \mathbf{V}_n^*(x) = \mathbf{D}_x^a U_n(s) - \mathbf{D}_x^a U_{n-1}(s)$ where s=2x-1 …(22)

$\mathbf{D}_x^a \mathbf{W}_n^*(x) = \mathbf{D}_x^a U_n(s) + \mathbf{D}_x^a U_{n-1}(s)$ where s=2x-1 …(23)

Where $U_0(s)$ and $U_1(s)$ from eq's(17) respectively.

REMARK 6-1: If $\alpha = 0.5$ and using properties of operator (D_x^a) eq's(*c* and *d*) then the general forms of fractional derivative of shifted Chebyshev polynomials yield:

$\mathbf{D}_x^{0.5} \mathbf{T}_0^*(\mathbf{x}) = \dfrac{1}{sqrt(p\ s)}$; where s=2x-1 …(24)

$\mathbf{D}_x^{0.5} \mathbf{U}_0^*(\mathbf{x}) = \dfrac{1}{sqrt(p\ s)}$; where s=2x-1 … (25)

7- System of Linear Fractional Integro-Defferintial Equation of Fredholm type (S.LFFIDE's): This system is given by the form [6]:

$$D_x^a f^i(x) = g^i(x) + \sum_{j=1}^{m} \int_a^b k_{ij}(x,t) f^j(t)dt$$

i=1,2,…,m ; a ≤ x ≤ b …(26)

The system has (*m*) independed equations, Any equation has (*m*) unknown functions $f^i(x)$. For example let ***m=2*** the system eq(26) gives two the following equations:

$$D_x^a f^1(x) = g^1(x) + \int_a^b \{k_{11}(x,t) f^1(t) + k_{12}(x,t) f^2(t)\}dt$$

$$D_x^a f^2(x) = g^2(x) + \int_a^b \{k_{21}(x,t) f^1(t) + k_{22}(x,t) f^2(t)\}dt$$

8- Solving the system of (S.LFFIDE's): In this section we give a numerical method to solve this system (i.e find unknown functions $f^i(x)$) by using two steps:

<u>Step1:</u> We approximate unknown functions by (***N***) known functions (Φ_r) by the form:

$$f^i(x) \approx f_N^i(x) = \sum_{r=0}^{N} c_r^i \Phi_r$$

…(27)

The important conditions (the known functions (bases) Φ_r must be linearly independed) so that in this work we choose orthogonal functions $(T_n^*, U_n^*, V_n^*$ ***and*** $W_n^*)$ as a bases functions.

Butting eq(27) in eq(26) yields eq (28), that is to give a system of (***M***) equations with ***(m(N+1))*** unknown parameters (c_r^i) , we must found them to find solution of the system.

$$D_x^a \sum_{r=0}^{N} c_r^i \Phi_r^i(x) = g^i(x) + \sum_{j=1}^{m} \int_a^b k_{ij}(x,t) \sum_{r=0}^{N} c_r^j \Phi_r^j(t)dt$$

i=1,2,…,m ; a ≤ x ≤ b …(28)

<u>Step2:</u> In this step we choose (***N***) points (x_k) in the interval [a,b] i.e$(a \le x_0 < x_1 < \ldots < x_N \le b)$, putting these points in the system equations (28) we have system eq(29) which gives ***(m(N+1))***independed equations with ***(m(N+1))*** unknown parameters (c_r^i),solve this system using Gauses-Elimination.

$$\sum_{r=0}^{N} c_r^i D_x^a \Phi_r^i(x_k) - \sum_{j=1}^{m}\sum_{r=0}^{N} c_r^j \int_a^b K_{ij}(x,t)\Phi_r^j(t_k)dt = g^i(x_k)$$

i=1,2,…,M ; k=0,1,…,N …(29)

Remark8-1: We can write the system of eq(29) by a matrix from as:

AC = G … (30)

where A is ***(m(N+1))*** by ***(m(N+1))*** matrix contains (***m*** by ***m***) sub-matrices (A_{ij}^{rk}) where I,j=1,2,..,m ; r,k=0,1,2,…,N we can write as:

$$A=\begin{bmatrix} A_{11}^{rk} & A_{12}^{rk} & \dots & A_{1m}^{rk} \\ A_{21}^{rk} & A_{22}^{rk} & \dots & A_{2m}^{rk} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ A_{m1}^{rk} & A_{m2}^{rk} & \dots & A_{mm}^{rk} \end{bmatrix} \quad \text{where}$$

r,k = 0,1,2,…,N …(31)

Where the sub-matrices are:

$$A_{ij}=\begin{bmatrix} a_{ij}^{00} & a_{ij}^{01} & \dots & a_{ij}^{0N} \\ a_{ij}^{10} & a_{ij}^{11} & \dots & a_{ij}^{1N} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ a_{ij}^{N0} & a_{ij}^{N1} & \dots & a_{ij}^{NN} \end{bmatrix} \quad \text{where}$$

the elements (a_{ij}^{rk}) from eq(33) …(32)

$$(a_{ij}^{rk})=\begin{cases} D_x^{a}\Phi_r(x_k)-\int_a^b k_{ii}(x_k,t)\Phi_k(x_k)dt & \text{if } i=j \\ -\int_a^b k_{ij}(x_k,t)\Phi_k(x_k)dt & \text{if } i\neq j \end{cases} \quad \dots(33)$$

The ***C*** column has m sub-columns (C_r^i) and we can write them as:

$$C=\begin{bmatrix} C_r^1 \\ C_r^2 \\ . \\ . \\ . \\ C_r^m \end{bmatrix} \quad \text{where}$$

$$C_r^i=\begin{bmatrix} c_0^i \\ c_1^i \\ . \\ . \\ . \\ c_N^i \end{bmatrix} \quad \dots(34)$$

Similarly, we can write the ***G*** column by ***(N+1)*** sub columns as:

$$G=\begin{bmatrix} g_k^1 \\ g_k^2 \\ . \\ . \\ . \\ g_k^m \end{bmatrix} \quad \text{where}$$

$$g_k^i=\begin{bmatrix} g_0^i \\ g_1^i \\ . \\ . \\ . \\ g_N^i \end{bmatrix} \quad \text{where } g_k^i = g^i(x_k) \quad \dots(35)$$

Remark 8-2: Since eq(33) and properties (***d*** and ***e***) of operator ***(*** $D_x^{0.5}$ ***)*** then all parts of matrix (***A***) has the part ($\frac{1}{\sqrt{p\,x}}$) so that we must choose points (x_k , k=0,1,2,…,N)s.t($2x_k$ -1> 0) or (x_k > 0.5), in general x_k= 0.55+k(0.45/N) …(36)

For example if N=2 then k=0,1,2 and x_k={0.55,0.77,1}.

Algorithm: Steps to solve (S.LFFIDE's):

Step1: Choose points x_k use eq(36) where k=0,1,…,N.

Step2: Evaluate (a_{ij}^{rk}) use eq's(33) and eq's (16-19, 20-23) for all i,j=1,2,…,m ;
k,r=0,1,2,…,N .

Step3: compute g_k^i use eq(35) for all i=1,2,…,m ; k=0,1,…,N .

Step4: construct the general matrix ***A*** use eq's(31-32) with step2 and column ***G*** using eq(35) with step3.

Step5: solve the system ***AC = G*** use Gauses-Elimination to find column ***C*** where

C =A\G or C=inv(A)*G.

Step6: find the approximate solutions $f_N^i(x)$ use eq(27) where the bases functions Φ_r are one of bases (T_n^*,U_n^*,V_n^* and W_n^*).

Note: in this work we use programs in MATLAB with six steps above to find approximate solution where:

F$_N$iT* means programs use steps above to find approximate solution to $f^i(x)$.

F$_N$iU* means programs use steps above to find approximate solution to $f^i(x)$.

F$_N$iV* means programs use steps above to find approximate solution to $f^i(x)$.

F$_N$iW* means programs use steps above to find approximate solution to $f^i(x)$.

<u>Example1</u>: Solve the linear system (FFIDE's):

$$D_x^{0.5} f^i(x) = g^i(x) + \sum_{j=1}^{2}\int_0^1 k_{ij}(x,t) f^j(t)dt$$

where k_{ij}(x,t)=1; I,j=1,2 and

$$g^1(x) = \frac{2x}{\sqrt{px}} - 1 \quad ;$$

$$g^2(x) = \frac{4x}{\sqrt{px}} - 0.5$$ with exact solution f^1(x)=x ; f^2(x)=2x.

<u>Solution</u>: Choose N=1 and running four programs above the results of this example listed in tables (1-1), (1-2) which obtained by using bases (T_n^*,U_n^*,V_n^* and W_n^*).

<u>Example 2</u>: Solve the linear system (FFIDE's):

$$D =_x^{0.5} f^i(x) = g^i(x) + \sum_{j=1}^{2}\int_0^1 k_{ij}(x,t) f^j(t)dt$$

where k_{ij}(x,t)=1; i,j=1,2 and

$$g^1(x) = \frac{2x}{\sqrt{px}} - 3 \quad ;$$

$$g^2(x) = \frac{1}{\sqrt{px}} + \frac{4x}{\sqrt{px}} + \frac{8x^2}{\sqrt{px}} - 0.5$$

with exact solution f^1(x)=x ; f^2(x)=1+2x+3x^3.

<u>Solution</u>: Choose N=2 **:** and running four programs above the results of this example listed in tables (2-1), (2-2) which obtained by using bases (T_n^*,U_n^*,V_n^* and W_n^*).

<u>Example 3:</u> Solve the linear system (FFIDE's):

$$D_x^{0.5} f^1(x) = g^1(x) + \int_0^1 f^2(t)dt \quad ;$$

$$D_x^{0.5} f^2(x) = g^2(x) + \int_0^1 f^1(t)dt$$

$$g^1(x) = \frac{1}{\sqrt{px}} + e^x erf(\sqrt{x}) - (e^3 - e^2)$$

;

$$g^2(x)=\frac{1}{\sqrt{px}}+e^x erf(\sqrt{x})-(e-1)$$

with exact solution $f^1(x)= e^x$; $f^2(x)= e^{x+2}$.

Solution: Choose N=5,8 and running four programs above the results of this example listed in tables (3-1), (3-2) which obtained by using bases (T^*_n, U^*_n, V^*_n and W^*_n).

Conclusions

1. All kinds of shifted Chebyshev polynomials (T^*_n, U^*_n, V^*_n and W^*_n) as a basis function which are used in this paper have proved their effectiveness in solving linear system (FFIDE's) numerically and finding accurate results, when unknown functions are algebraic functions, other functions the U^*_n gives better solution than other bases.
2. All Chebyshev polynomials depends on N as N increasing, the error term is decreased.
3. The results show a marked improvement in the least square errors from which we conclude that.

References

[1]-Adolfsson,K.& Enelund, M.,"Dicretization of Integro-Differential Equations Modeling Dynamic Fractional order Viscoelasticity"Appl.Mech.,2004.

[2]-Atkinson, K.E.,"An Introduction to Numerical Analysis",Second Edition John Wiley and Sons, Inc.,New York,1989.

[3]-Chebyshev Polynomials ,J.C.MASON,D.C.HANDSCOMB,2003 by CRC Press LLC.

[4]-Del-Castillo,D.,"Fractional calculus :Basic Theory and Applications (part1)",Lectures presented at the Institute of Mathematics, City.Mexico, UNAM.August 2005.

[5]- Diethelm,K,& Free,A.D.,"The frac PECE subroutine for the Numerical solution of Differential Equations of Fractional Order",In S.Heinzel,T.Plesser (eds):For schung and wissens chaftliches Rechnen:Beitrage zum Heinz-Billing,-preis 1998,Gotting,pp.57-71,1999.

[6]- Gorenflo,R.&Mainardi,F,"Fractional calculus:Integral and Differential Equations of Fractional Order",Spriner Verlag,Wien and NewYork,1997.

[7]-Jerri,A.J.,"Introduction to Integral Equationswith.Applications".MarcelDekker,Inc.,New York and Basel,1985.

[8]-Oldham.K.B.& spanier J.,"The Fractional calculus :Theory and Applications of Differentiation and Integration.to Arbitrary order",Academic press,New York and London,1974.

[9]-Ross,B.,"Fractional calculus and Its Applications", Lecture Notes in Mathematics vol.457,proceedings of the International conference Held at the University of New Haven,June 1974,New York,1975.

[10]-Schmidt A.& Gaul L.,"FE Implementation of viscoelastic Constitutive stress-strain Relations Involving Fractional Time Derivatives", A Report, Institute A Fur Mechanik, University stuttart Germany,2000.

Ex1-(N1)

$f1=x$

X_i	F1=x	F_N1T	F_N1U	F_N1V	F_N1W
0.1	0.1	0.1	0.1	0.1	0.1
0.2	0.2	0.2	0.2	0.2	0.2
0.3	0.3	0.3	0.3	0.3	0.3
0.4	0.4	0.4	0.4	0.4	0.4
0.5	0.5	0.5	0.5	0.5	0.5
0.6	0.6	0.6	0.6	0.6	0.6
0.7	0.7	0.7	0.7	0.7	0.7
0.8	0.8	0.8	0.8	0.8	0.8
0.9	0.9	0.9	0.9	0.9	0.9
1	1	1	1	1	1
L.S. Erorr		0.0	0.0	0.0	0.0

Table(1-1)

$f2=2x$

X_i	F2=2x	F_N2T	F_N2U	F_N2V	F_N2W
0.1	0.2	0.2	0.2	0.2	0.2
0.2	0.4	0.4	0.4	0.4	0.4
0.3	0.6	0.6	0.6	0.6	0.6
0.4	0.8	0.8	0.8	0.8	0.8
0.5	1	1	1	1	1
0.6	1.2	1.2	1.2	1.2	1.2
0.7	1.4	1.4	1.4	1.4	1.4
0.8	1.6	1.6	1.6	1.6	1.6
0.9	1.8	1.8	1.8	1.8	1.8
1	2	2	2	2	2
L.S. Erorr		0.0	0.0	0.0	0.0

Table(1-2)

Ex2:

f1=x

X	F1=x	F_n1T	F_n1U	F_n1V	F_n1W
0.1	0.1	9.999999999993704e-002	9.999999999999347e-02	9.999999999998016e-002	9.999999999998777e-002
0.2	0.2	1.999999999999305e-001	1.999999999999929e-01	1.999999999999784e-001	1.999999999999863e-001
0.3	0.3	2.999999999999242e-01	2.999999999999924e-01	2.999999999999767e-001	2.999999999999849e-001
0.4	0.4	3.999999999999181e-01	3.999999999999919e-01	3.999999999999750e-001	3.999999999999834e-001
0.5	0.5	4.999999999999122e-01	4.999999999999914e-01	4.999999999999734e-001	4.999999999999821e-001
0.6	0.6	5.999999999999065e-01	5.999999999999909e-01	5.999999999999718e-001	5.999999999999808e-001
0.7	0.7	6.999999999999010e-01	6.999999999999903e-01	6.999999999999702e-001	6.999999999999796e-001
0.8	0.8	7.999999999998956e-01	7.999999999999898e001	7.999999999999687e-001	7.999999999999785e-001
0.9	0.9	8.999999999998904e-01	8.999999999999894e-01	8.999999999999672e-001	8.999999999999774e-001
1	1	9.999999999998853e-01	9.999999999999889e-01	9.999999999999657e-001	9.999999999999764e-001
L.S. Error		8.356225665870656e-026	8.147801643626350e-028	7.665109533443739e-027	3.472446447646524e-027

Table(2-1)

f2=1+2x+3x^2

X_i	F2= 1+2x+3x^2	F_n2T	F_n2U	F_n2V	F_n2W
0.1	1.23	1.229999999999920e+000	1.229999999999992e+000	1.229999999999976e+000	1.229999999999982e+000
0.2	1.52	1.519999999999917e+000	1.519999999999992e+000	1.519999999999975e+000	1.519999999999982e+000
0.3	1.87	1.869999999999914e+000	1.869999999999992e+000	1.869999999999974e+000	1.869999999999982e+000
0.4	2.28	2.279999999999911e+000	2.279999999999991e+000	2.279999999999973e+000	2.279999999999981e+000
0.5	2.75	2.749999999999907e+000	2.749999999999991e+000	2.749999999999973e+000	2.749999999999981e+000
0.6	3.28	3.279999999999903e+000	3.279999999999991e+000	3.279999999999971e+000	3.279999999999980e+000
0.7	3.87	3.869999999999899e+000	3.869999999999990e+000	3.869999999999970e+000	3.869999999999979e+000
0.8	4.52	4.519999999999896e+000	4.519999999999990e+000	4.519999999999969e+000	4.519999999999978e+000
0.9	5.23	5.229999999999891e+000	5.229999999999990e+000	5.229999999999968e+000	5.229999999999977e+000
1	6.00	5.999999999999886e+000	5.999999999999989e+000	5.999999999999965e+000	5.999999999999976e+000
L.S. Error		9.253832949302878e-026	8.664819360241697e-028	8.171885548270880e-027	4.025995491859953e-027

Table (2-2)

Ex3-(N=5)

$f1=e^x$

X_i	$F1=e^x$	$F_{\pi}1T$	$F_{\pi}1U$	$F_{\pi}1V$	$F_{\pi}1W$
0.1	1.1052	1.217173618485968e+000	1.217173618485225e+000	1.217173618484390e+000	1.217173618485821e+000
0.2	1.2214	1.348808726382611e+000	1.348808726381767e+000	1.348808726380817e+000	1.348808726382442e+000
0.3	1.3499	1.492138425910013e+000	1.492138425909072e+000	1.492138425908013e+000	1.492138425909823e+000
0.4	1.4918	1.648301689258319e+000	1.648301689257285e+000	1.648301689256121e+000	1.648301689258109e+000
0.5	1.6487	1.818649813975645e+000	1.818649813974523e+000	1.818649813973262e+000	1.818649813975416e+000
0.6	1.8221	2.004746422968084e+000	2.004746422966880e+000	2.004746422965525e+000	2.004746422967838e+000
0.7	2.0138	2.208367464499703e+000	2.208367464498420e+000	2.208367464496977e+000	2.208367464499440e+000
0.8	2.2255	2.431501212192541e+000	2.431501212191184e+000	2.431501212189658e+000	2.431501212192263e+000
0.9	2.4596	2.676348265026613e+000	2.676348265025184e+000	2.676348265023579e+000	2.676348265026320e+000
1	2.7183	2.945321547339905e+000	2.945321547338409e+00	2.945321547336728e+000	2.945321547339598e+000
L.S. Error		3.145498694500295e-001	3.145498694458787e-001	3.145498694412111e-001	3.145498694491827e-001

Table (3-1)

e^{x+2}

X_i	$F2=e^{x+2}$	$F_{\pi}2T$	$F_{\pi}2U$	$F_{\pi}2V$	$F_{\pi}2W$
0.1	8.1662	8.344570670485336e+000	8.344570670484179e+000	8.344570670482883e+000	8.344570670485091e+000
0.2	9.0250	9.197138861722015e+000	9.197138861720880e+000	9.197138861719605e+000	9.197138861721777e+000
0.3	9.9742	1.014661385324100e+001	1.014661385323986e+001	1.014661385323857e+001	1.014661385324076e+001
0.4	11.0232	1.119996602222405e+001	1.119996602222288e+001	1.119996602222156e+001	1.119996602222380e+001
0.5	12.1825	1.236588185161165e+001	1.236588185161044e+001	1.236588185160907e+001	1.236588185161140e+001
0.6	13.4637	1.365476393010303e+001	1.365476393010177e+001	1.365476393010035e+001	1.365476393010277e+001
0.7	14.8797	1.507873095215615e+001	1.507873095215484e+001	1.507873095215336e+001	1.507873095215588e+001
0.8	16.4446	1.665161771798768e+001	1.665161771798632e+001	1.665161771798478e+001	1.665161771798740e+001
0.9	18.1741	1.838897513357305e+001	1.838897513357164e+001	1.838897513357004e+001	1.838897513357276e+001
1	20.0855	2.030807021064640e+001	2.030807021064493e+001	2.030807021064328e+001	2.030807021064610e+001
L.S. Error		3.707334532658249e-001	3.707334532609456e-001	3.707334532554368e-001	3.707334532648115e-001

Table (3-2)

Ex3-(N=8)

$f2 = e^{x+2}$

X_i	$F2=e^{x+2}$	$F_N 2T$	$F_N 2U$	$F_N 2V$	$F_N 2W$
0.1	8.1662	8.165948783324557e+000	8.165948783324557e+000	8.165948782393510e+000	8.165948783324557e+000
0.2	9.0250	9.024811590257933e+000	9.024811590257933e+000	9.024811589400859e+000	9.024811590257933e+000
0.3	9.9742	9.973982603730176e+000	9.973982603730176e+000	9.973982602882709e+000	9.973982603730176e+000
0.4	11.0232	1.102297143739168e+001	1.102297143739168e+001	1.102297143652431e+001	1.102297143739168e+001
0.5	12.1825	1.218228122197784e+001	1.218228122197784e+001	1.218228122107844e+001	1.218228122197784e+001
0.6	13.4637	1.346351658439830e+001	1.346351658439830e+001	1.346351658346254e+001	1.346351658439830e+001
0.7	14.8797	1.487950131679788e+001	1.487950131679788e+001	1.487950131582455e+001	1.487950131679788e+001
0.8	16.4446	1.644440741892184e+001	1.644440741892184e+001	1.644440741791092e+001	1.644440741892184e+001
0.9	18.1741	1.817389719711830e+001	1.817389719711830e+001	1.817389719607026e+001	1.817389719711830e+001
1	20.0855	2.008528010331039e+001	2.008528010331039e+001	2.008528010222593e+001	2.008528010331039e+001
L.S. Error		4.456612083042770e-007	4.456612083042770e-007	4.456651790039209e-007	4.456612083042770e-007

Table (4-2)

$f1 = e^{x}$

X_i	$F1=e^{x}$	$F_N 1T$	$F_N 1U$	$F_N 1V$	$F_N 1W$
0.1	1.1052	1.105050268611320e+000	1.105050268611320e+000	1.105050268102487e+000	1.105050268611320e+000
0.2	1.2214	1.221261937817951e+000	1.221261937817951e+000	1.221261937223005e+000	1.221261937817951e+000
0.3	1.3499	1.349698604926968e+000	1.349698604926968e+000	1.349698604250336e+000	1.349698604926968e+000
0.4	1.4918	1.491646427429108e+000	1.491646427429108e+000	1.491646426676433e+000	1.491646427429108e+000
0.5	1.6487	1.648526276731655e+000	1.648526276731655e+000	1.648526275908515e+000	1.648526276731655e+000
0.6	1.8221	1.821908273662720e+000	1.821908273662720e+000	1.821908272774079e+000	1.821908273662720e+000
0.7	2.0138	2.013527655690326e+000	2.013527655690326e+000	2.013527654740413e+000	2.013527655690326e+000
0.8	2.2255	2.225302200197914e+000	2.225302200197914e+000	2.225302199190300e+000	2.225302200197914e+000
0.9	2.4596	2.459351428157911e+000	2.459351428157911e+000	2.459351427095637e+000	2.459351428157911e+000
1	2.7183	2.718017811254497 1e+000	2.718017811254497 1e+000	2.718017811430656e+000	2.718017811254497 1e+000
L.S. Error		4.272778323838178e-007	4.272778323838178e-007	4.272813525296136e-007	4.272778323838178e-007

Table (4-1)