Fully $\overline{P}$ - $P$ – Stable Rings

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Abstract

M.S.Abbas [1] introduced and studied the concept of fully stable $R$-modules and called a ring $R$ is fully stable (pseudo - stable) if it is fully stable ( pseudo - stable ) $R$-module. And A.M.Abdul-Daim [2] introduced and studied the concept of fully $\overline{P}$ - stable rings as a generalization of fully stable rings.

The purpose of this paper is to generalize the concept of fully pseudo – stable rings to fully $\overline{P}$ – pseudo - stable rings. Some properties and characterizations of fully $\overline{P}$ – pseudo – stable rings are obtained. A condition is given such that a fully $\overline{P}$ – pseudo – stable ring is fully $\overline{P}$ – stable.

Introduction

In this paper , $R$ represents a commutative ring with identity and all modules are left unitary.

M.S.Abbas [1] was introduced the concept of a fully stable $R$-module and then introduced the concept of a fully pseudo-stable (fully $p$-stable) module as a generalization of a fully stable module.

(1) Definition

An $R$-module $M$ is said to be fully stable module, if for each $R$-homomorphism $\alpha:N \rightarrow M$ of any submodule $N$ of $M$ into $M$, we have $\alpha (N) \subseteq N$ . A ring $R$ is fully stable if it is a fully stable $R$- module.

(2) Definition

An $R$-module $M$ is said to be fully $p$ - stable if for each $R$-
monomorphism \( \alpha : N \to M \) of any submodule \( N \) of \( M \) into \( M \), we have \( \alpha (N) \subseteq N \). A ring \( R \) is fully pseudo stable (fully \( p \)-stable) if and only if it is a fully \( p \)-stable \( R \)-module [1].

In [2] the concept of a \( \pi \)-stable ring is investigated which includes the class of fully stable rings and that of \( \pi \)-regular rings.

(3) Definition
A ring \( R \) is called fully \( \pi \)-stable if and only if for each element of \( x \) in \( R \), there exists a positive integer \( n \) such that for every \( R \)-homomorphism \( \alpha : Rx^n \to R \) we have \( \alpha (Rx^n) \subseteq Rx^n \).

In an analogous manner, we introduce now a class of rings larger than the class of fully \( \pi \)-stable rings.

(4) Definition
Let \( R \) be any ring. An element \( x \) in \( R \) is called \( \pi \)-pseudo-stable (abbreviated \( \pi \)-p-stable) if there exists a positive integer \( n \) such that for every \( R \)-homomorphism \( \alpha : Rx^n \to R \) we have \( \alpha (Rx^n) \subseteq Rx^n \).

A ring \( R \) is called fully \( \pi \)-pseudo-stable if and only if every element in \( R \) is \( \pi \)-pseudo-stable.

It is clear that every \( \pi \)-stable element of an arbitrary ring is \( \pi \)-p-stable. Hence every fully \( \pi \)-stable rings is fully \( \pi \)-p-stable, we conjecture the converse is not true, but we recall that a non zero \( R \)-module \( M \) is said to be uniform if each non zero submodules of \( M \) has non zero intersection with every non zero submodule of \( M \). A ring \( R \) is uniform if it is uniform \( R \)-module, then we have the following:

(5) Proposition
Every fully \( \pi \)-p-stable uniform ring is fully \( \pi \)-stable ring.

Proof
Let \( R \) be a fully \( \pi \)-p-stable uniform ring. For any element \( x \) in \( R \) there exists a positive integer \( n \) and for every \( R \)-homomorphism \( f : Rx^n \to R \). If \( \ker(f) = (0) \), there is nothing to prove. Otherwise, let \( y \in \ker(f) \cap \ker(I_{Rx^n} + f) \). Then \( f(y) = 0 \) and \( (I_{Rx^n} + f)(y) = 0 \). Now, \( y = y + f(y) = (I_{Rx^n} + f)(y) = 0 \). Thus \( \ker(f) \cap \ker(I_{Rx^n} + f) = (0) \), but \( R \) is uniform, hence \( \ker(I_{Rx^n} + f) = (0) \), that is, \((I_{Rx^n} + f) : Rx^n \to R \) is an \( R \)-monomorphism. Since \( R \) fully \( \pi \)-p-stable, then \((I_{Rx^n} + f)(Rx^n) \subseteq Rx^n \), hence \( f(Rx^n) \subseteq Rx^n \).

W. D. Weakly [3] was introduced the concept of terse module. An \( R \)-module is said to be terse iff distinct submodules are not isomorphic. He proved that for an \( R \)-module to be terse, it is enough to have the property that distinct cyclic submodules are not isomorphic.

A ring \( R \) is terse if and only if it is terse \( R \)-module. The following is a generalization for terse rings.

(6) Definition
A ring \( R \) is called \( \pi \)-terse iff for any two elements \( x \) and \( y \) in \( R \) there exists a positive integer \( n \) such that if \( Rx^n \neq Ry \) implies \( Rx^n \nsubseteq Ry \).

In the following proposition we show that the concepts of a \( \pi \) - terseness and full \( \pi \)-p-stability are coincide.

(7) Proposition
A ring \( R \) is \( \pi \)-terse if and only if it is fully \( \pi \)-p-stable ring.
Proof

Suppose that $R$ is \(\pi\)–terse ring and there exists an element \(x\) in \(R\) and \(R\)-monomorphism \(f:R_x^n \rightarrow R\) such that \(f(R_x^n) \subset Rx^n\) for each positive integer \(n\), then \(Rx^n\) and \(f(R_x^n) = R_{f(x)}\) are two distinct ideals of \(R\). Since \(R\) is \(\pi\)–terse ring, then \(R_{f(x)} = f(R_x^n)\) is not isomorphic to \(Rx^n\) which is not true. Hence \(R\) is fully \(\pi\)–\(p\)-stable ring.

Conversely, suppose that \(R\) is a fully \(\pi\)–\(p\)-stable ring and \(R\) has two elements \(x\) and \(y\) such that \(Rx_x^n \subseteq Ry\) but \(Rx_x^n \neq Ry\) for each positive integer \(t\). We can assume that \(Rx_x^n \subset Ne\). Then there exists a non–zero element \(z\) in \(Rx_x^n\) which is not in \(Ry\). Let \(f:Rx_x^n \rightarrow Ry\) be an isomorphism, consider the following two \(R\)-monomorphisms, \(I_{Rx_x}^n \circ f : Rx_x^n \rightarrow R\) and \(I_{Rx_x}^n \circ f^1: Ry \rightarrow R\), since \(R\) is fully \(\pi\)–\(p\)-stable ring, then \((I_{Rx_x}^n \circ f)(Rx_x^n) \subseteq Rx_x^n\) and \((I_{Rx_x}^n \circ f^1)(Ry) \subseteq Ry\). Now, \(z = (I_{Rx_x}^n \circ f^1 \circ I_{Rx_x}^n \circ f)_z \in Ry\) which is a contradiction.

Proposition (7) together with proposition (5) give:

(8)Corollary

Let \(R\) be a uniform ring and \(\pi\)–terse ring, then \(R\) is fully \(\pi\)-stable ring.

From proposition (7) we see that every fully \(\pi\)-stable ring, is \(\pi\)–terse, hence we have the following proposition:

(9) Proposition

Let \(R\) be a fully \(\pi\)–stable ring and let \(x\) and \(y\) be any two elements in \(R\) with \(Ry\) a direct summand of \(R\) then there exists a positive integer \(n\) such that if \(Rx^n\) is isomorphic to \(Ry\), then \(Rx^n\) is direct summand of \(R\).

Proof

Since \(R\) is fully \(\pi\)–stable ring, then \(R\) is \(\pi\)–terse, so if \(Rx^n \cong Ry\), then \(Rx^n = Ry\), which implies that \(Ry\) is a direct summand of \(R\).

Next, we will characterize fully \(\pi\)–stable rings among fully \(\pi\)-\(p\)-stable rings. However, we shall need the following lemma (for its proof, see[1]).

(10) Lemma

Let \(M\) be an \(R\)-module and \(I\) an ideal of \(R\). Then \(\text{ann}_M(I) \cong \text{Hom}_R(R/IM\).

(11)Theorem

Let \(R\) be a ring. Then the following statements are equivalent:

1. \(R\) is a fully \(\pi\)–stable ring.
2. \(R\) is a \(\pi\)–terse ring and for every element \(x\) in \(R\) there exists a positive integer \(n\) such that \(Rx^n \cong \text{Hom}_R(Rx^n, R)\).

Proof

(1) implies (2). Assume that \(R\) is a fully \(\pi\)-stable ring, then \(R\) is \(\pi\)-terse. Since \(R\) is fully \(\pi\)-stable ring, then for every element \(x\) in \(R\) there exists a positive integer \(n\) such that \(Rx^n = \text{ann}(\text{ann}(Rx^n))\) [2]. By Lemma (10) \(\text{ann}(\text{ann}(Rx^n)) \cong \text{Hom}_R(R/\text{ann}(Rx^n), R) = \text{Hom}(Rx^n, R)\) which implies that \(Rx^n \cong \text{Hom}(Rx^n, R)\).

(2) implies (1). Suppose that \(R\) is \(\pi\)–terse and for every element \(x\) in \(R\) there exists a positive integer \(n\) such that \(Rx^n \cong \text{Hom}(Rx^n, R)\). By Lemma (10) \(\text{ann}(\text{ann}(Rx^n)) \cong \text{Hom}(R/\text{ann}(Rx^n), R) \cong \text{Hom}(Rx^n, R)\), then \(Rx^n \cong \text{ann}(\text{ann}(Rx^n))\) is \(\pi\)-terse of \(R\) implies that \(Rx^n = \text{ann}(\text{ann}(Rx^n))\). Hence \(R\) is fully \(\pi\)-stable ring.
The following corollary follows from proposition (7) which gives a characterization of fully $\pi$-stable rings among fully $\pi-p$-stable rings.

**Corollary**

The following statements are equivalent for a ring $R$.

1) $R$ is a fully $\pi$-stable ring.
2) $R$ is a fully $\pi-p$-stable ring and for every element $x$ in $R$ there exists a positive integer $n$ such that $Rx^n \cong \text{Hom}(Rx^n, R)$.

**Discussion**

From all the above we have the following:

1) Every fully $\pi$-stable ring is fully $\pi-p$-stable.
2) Every fully $\pi-p$-stable uniform ring is fully $\pi$-stable ring.

3) A ring $R$ is $\pi$-terse if and only if it is fully $\pi-p$-stable.
4) A ring $R$ is fully $\pi$-stable ring if and only if $R$ is fully $\pi-p$-stable ring and for every element $x$ in $R$ there exists a positive integer $n$ such that $Rx^n \cong \text{Hom}_R(Rx^n, R)$.

**References**