Fuzzy Hilbert Spaces

Abstract
we introduce the definition of a fuzzy inner product space and discuss some properties of this space, and we use the definition of fuzzy inner product space to introduced anew definitions such that the definition of fuzzy Hilbert space, Fuzzy convergence, Fuzzy complete, and we studied the relation between ordinary inner product space and fuzzy inner product space.

Keywords: fuzzy set, fuzzy inner product space

1-Introduction
J.R. Kider was introduced the definition of fuzzy inner product [2], which is using to study the fuzzy inner product spaces in detail and these gave us new results.

we use the definition of fuzzy inner product space appeared in [2] to study this space and investigate some properties of this space, we prove that for any fuzzy inner product space $H$ there exists a fuzzy Hilbert space $Y$ and an isomorphism $T$ from $H$ onto fuzzy dense subspace of $Y$. The space $Y$ is unique except for isomorphism's (See theorem 3.12).

2- Basic concepts:
Definition 1.1:
Let $X$ be any set of element. A fuzzy set $\tilde{A}$ in $X$ is characterized by a membership function, $\mu_A(x): X \rightarrow I$, where $I$ is the closed interval [0,1]. Then we can write a fuzzy set $\tilde{A}$ as:

$\tilde{A} = \{(x, \mu_A(x)) | x \in X, 0 \leq \mu_A(x) \leq 1\}$

The collection of all fuzzy subsets of $X$ will be denoted by $l^X$, i.e.,

$l^X = \{A: A$ is fuzzy subset of $X\}$

Remarks 1.2:
Following, some fundamental concept related to the basic operations and concepts of fuzzy subsets of $X$.

Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy subsets of $X$ with membership functions $\mu_A$ and $\mu_B$ respectively, then:

1. $\tilde{A} \subseteq \tilde{B}$ if and only if $\mu_A(x) \leq \mu_B(x)$, $\forall x \in X$
2- \( \hat{A} = \hat{B} \) if and only if \( \mu_\hat{A}(x) = \mu_\hat{B}(x), \forall x \in X. \)

3- The complement of \( \hat{A} \) (denoted by \( \hat{A}^c \)) is also a fuzzy set which has the membership function \( \mu_{\hat{A}^c}(x) = 1 - \mu_\hat{A}(x). \)

4- \( \hat{A} = \emptyset \) if and only if \( \mu_\hat{A}(x) = 0, \forall x \in X, \) where \( \emptyset \) is the empty set.

5- The height of fuzzy set is the supremum value of \( \mu_\hat{A}(x) \) over all \( x \in X. \) If the height is 1, then \( \hat{A} \) is normal, otherwise it is subnormal.

6- The support of \( \hat{A} \) is the set of all elements \( x \in X \) at which \( 0 < \mu_\hat{A}(x) \leq 1 \) and is denoted by \( \text{supp}(\hat{A}). \)

7- A point \( x \in X \) is said to be a crossover point of \( \hat{A} \) if \( \mu_\hat{A}(x) = 0.5. \)

8- \( \hat{C} = \hat{A} \cap \hat{B} \) is a fuzzy set with membership function \( \mu_{\hat{C}}(x) = \min \{ \mu_{\hat{A}}(x), \mu_{\hat{B}}(x) \}. \)

9- \( \hat{D} = \hat{A} \cup \hat{B} \) is a fuzzy set with membership function \( \mu_{\hat{D}}(x) = \max \{ \mu_{\hat{A}}(x), \mu_{\hat{B}}(x) \}. \)

10- If \( \left( \mu_{\hat{A}}(x) = 0, \forall x \in X \right) \) then \( \hat{A} \) and \( \hat{B} \) are said to be separated fuzzy sets.

11- The product of \( \hat{A} \) and \( \hat{B} \) (denoted by \( \hat{A} \times \hat{B} \)) is also fuzzy set which have a membership function \( \mu_{\hat{A} \times \hat{B}}(x) = \mu_{\hat{A}}(x) \mu_{\hat{B}}(x). \)

12- n-th power of \( \hat{A} \) (denoted by \( \hat{A}^n \)) is also a fuzzy set which have a membership function \( \mu_{\hat{A}^n}(x) = [\mu_{\hat{A}}(x)]^n \) where \( n \) is a positive integer and consequently \( \hat{A} \subseteq \hat{A}^n \) for all \( n \geq m \geq 0. \)

**Theorem 1.3:**

Let \( \{ \hat{A}_j \}_{j \in J} \) be a collection of fuzzy sets in \( X. \)

(a) If \( \hat{A}_i \subseteq \hat{A}_j \) then:

(i) \( \hat{A}_i \cap \hat{A}_j = \hat{A}_i \)

(ii) \( \hat{A}_i \cup \hat{A}_j = \hat{A}_j \)

(iii) \( \hat{A}_i \supseteq \hat{A}_j \) where \( i, j \in J \)

(b) Commutative laws: For \( i, j \in J \)

\[ \hat{A}_i \cup \hat{A}_j = \hat{A}_j \cup \hat{A}_i \]

(c) Associative laws: For \( i, j, k \in J \)

\[ \hat{A}_i \cup (\hat{A}_j \cup \hat{A}_k) = (\hat{A}_i \cup \hat{A}_j) \cup \hat{A}_k \]

\[ \hat{A}_i \cap (\hat{A}_j \cap \hat{A}_k) = (\hat{A}_i \cap \hat{A}_j) \cap \hat{A}_k \]

(d) Distributive laws: for \( i, j, k \in J \)

\[ \hat{A}_i \cup (\hat{A}_j \cap \hat{A}_k) = (\hat{A}_i \cup \hat{A}_j) \cap (\hat{A}_i \cup \hat{A}_k) \]

\[ \hat{A}_i \cap (\hat{A}_j \cup \hat{A}_k) = (\hat{A}_i \cap \hat{A}_j) \cup (\hat{A}_i \cap \hat{A}_k) \]

(e) De Morgan’s laws: For \( i, j \in J \)

\[ (\hat{A}_i \cap \hat{A}_j)^c = \hat{A}_i^c \cup \hat{A}_j^c \]

\[ (\hat{A}_i \cup \hat{A}_j)^c = \hat{A}_i^c \cap \hat{A}_j^c \]

**Definition 1.4:**

If \( \hat{A} \) and \( \hat{B} \) are fuzzy sets in non-empty set \( X \) respectively then the Cartesian product \( \hat{A} \times \hat{B} \).
\( \mathcal{A} \times \mathcal{B} \) of \( \mathcal{A} \) and \( \mathcal{B} \) is a fuzzy set with membership function
\[ p_{\mathcal{A} \times \mathcal{B}}(x,y) = \min \{ p_{\mathcal{A}}(x), p_{\mathcal{B}}(y) \} \text{ for all } (x,y) \text{ in } X \times Y. \]

**Definition 1.5:**
A fuzzy point \( p \) in \( X \) is a fuzzy set with membership function
\[ F(\alpha) = \begin{cases} \alpha, & \text{if } y = x \\ 0, & \text{otherwise} \end{cases} \]
for all \( y \) in \( X \), where \( 0 \leq \alpha < 1 \) is said to have support \( \alpha \) and value \( \alpha \).

We denote this fuzzy point by \( x_\alpha \) or \( (x, \alpha) \).

Two fuzzy points \( x_\alpha \) and \( y_\beta \) are said to be distinct if and only if \( x \neq y \).

**Remark 1.6**
Two fuzzy points \( x_\alpha \) and \( y_\beta \) are said to be equal if and only if \( x = y \) and \( \alpha = \beta \), where \( \alpha, \beta \in \{0, 1\} \).

**Definition 1.7**
Let \( f \) be a function from a nonempty set \( X \) to a nonempty set \( Y \).
If \( \mathcal{B} \) is a fuzzy set in \( Y \) then \( f^{-1}(\mathcal{B}) \) is a fuzzy set in \( X \) defined by:
\[ \mu_{f^{-1}(\mathcal{B})}(x) = \mu_{\mathcal{B}}(f(x)) \text{ for all } x \text{ in } X. \]
Equivalently
\[ \mu_{f^{-1}(\mathcal{B})}(x) = \sup \{ \mu_{\mathcal{B}}(y) : f(x) = y \}. \]

Also if \( \mathcal{A} \) is a fuzzy set in \( X \) then \( f(\mathcal{A}) \) is a fuzzy set in \( Y \) defined by
\[ \mu_{f(\mathcal{A})}(y) = \inf \{ \mu_{\mathcal{A}}(x) : x \in f^{-1}(y) \} = \begin{cases} 1, & \text{if } y = f(x) \\ 0, & \text{otherwise} \end{cases} \text{ for all } y \text{ in } Y. \]

**Definition 1.8**
Let \( x_\alpha \) be a fuzzy point and \( \mathcal{A} \) a fuzzy set in \( X \). Then \( x_\alpha \) is said to be in \( \mathcal{A} \) or belongs to \( \mathcal{A} \) denoted by \( x_\alpha \in \mathcal{A} \) if and only if \( \alpha \leq \mu_{\mathcal{A}}(x) \).

The set of all fuzzy points in \( X \) is denoted by \( P(X) \).

**Remark 1.9**
Let \( \mathcal{A} \) and \( \mathcal{B} \) be fuzzy sets in \( X \).
Then:
1. \( \mathcal{A} \subseteq \mathcal{B} \) if and only if for every \( x_\alpha \in \mathcal{A} \) implies \( x_\alpha \in \mathcal{B} \).
2. \( \mathcal{A} = \mathcal{B} \) if and only if \( x_\alpha \in \mathcal{A} \) if and only if \( x_\alpha \in \mathcal{B} \) for all \( x_\alpha \in P(X) \).
3. \( x_\alpha \in \mathcal{A} \cup \mathcal{B} \) if and only if \( x_\alpha \in \mathcal{A} \) or \( x_\alpha \in \mathcal{B} \).
4. \( x_\alpha \in \mathcal{A} \cap \mathcal{B} \) if and only if \( x_\alpha \in \mathcal{A} \) and \( x_\alpha \in \mathcal{B} \).

**Theorem 1.10**
Let \( \{ A_j \}_{j \in J} \) be a family of fuzzy sets in \( X \) where \( J \) is any index.
Then \( x_\alpha \in \bigcup_{j \in J} A_j \) if and only if \( x_\alpha \in A_k \) for some \( k \in J \).

**Proposition 1.11**
Let \( f : X \rightarrow Y \) be a function. Then for a fuzzy point \( x_\alpha \in X \), \( f(x_\alpha) \) is a fuzzy point in \( Y \) and \( |x_\alpha| = |f(x_\alpha)| \).

**Theorem 1.12**
A fuzzy set \( \mathcal{A} \) in \( X \) is a union of all its fuzzy points.

2- Fuzzy Inner Product Spaces:
Fuzzy inner product spaces are special fuzzy normed spaces as we shall see. In fact fuzzy inner product spaces are probably the most natural generalization of fuzzy Euclidean space and the reader should note the great harmony and beauty of the concepts.

**Definition 2.1:**
Let \( H \) be a vector space over field \( \mathbb{K} \), an inner product on \( H \) is a mapping \( \langle \cdot, \cdot \rangle : H \times H \rightarrow \mathbb{K} \) of \( H \) that is with every pair of vector \( x \) and
there is associated a scalar which is written \( \langle x, y \rangle \) and is called the inner product of \( x \) and \( y \) such that for all vectors \( x, y \) and scalar \( r \) we have:

\[
(IP_1)(x + z, y) = \langle x, y \rangle + \langle z, y \rangle \\
(IP_2)(rx, y) = r\langle x, y \rangle \\
(IP_3)(x, y) = \overline{\langle y, x \rangle} \\
(IP_4)(x, x) \geq 0 \text{ and } (x, x) = 0 \iff x = 0
\]

**Definition 2.2:**

A fuzzy inner product space on \( H \), where \( H \) is a vector space over the field \( \mathbb{K} \) (where \( \mathbb{K} \) is either \( \mathbb{R} \) or \( \mathbb{C} \)) is a mapping of \( H \times H \) into the field \( \mathbb{K} \), that is with every pair of fuzzy vectors \( x, y \) there is associated a scalar \( \lambda \) which is written

\[
\lambda = \text{mtn}[\alpha, \beta] \quad (\alpha, \beta \in (0, 1])
\]

and is called the fuzzy inner product of \( x \) and \( y \) such that for all fuzzy vectors \( x, y \) with

\[
(x, y) = \langle x, y \rangle (\lambda),
\]

we have:

\[
(FIP_1)(x + z, y)(\lambda) = \langle x, y \rangle (\lambda) + \langle z, y \rangle (\lambda) \\
(FIP_2)(rx, y)(\lambda) = r\langle x, y \rangle (\lambda) \\
(FIP_3)(x, y)(\lambda) = \overline{\langle y, x \rangle (\lambda)} \\
(FIP_4)(x, x)(\lambda) \geq 0 \text{ and } (x, x)(\lambda) = 0 \iff x = 0
\]

**Proposition 2.3:**

If \( (H, \langle \cdot, \cdot \rangle) \) is an ordinary inner product space then \( (H, \langle \cdot, \cdot \rangle) \) is a fuzzy inner product space, where \( \lambda = \text{mtn}[\alpha, \beta] \) and \( \alpha, \beta \in (0, 1] \).

**Proof:**

Let \( x, y \in H \) and \( r \in \mathbb{K} \) where \( \alpha, \beta, \sigma \in (0, 1] \) with \( \lambda = \text{mtn}[\alpha, \beta, \sigma] \) then

\[
(FIP_1)(x + z, y)(\lambda) = \frac{1}{\lambda} (x + z, y) = \frac{1}{\lambda} (x, y) + \frac{1}{\lambda} (z, y)
\]

\[
(FIP_2)(rx, y)(\lambda) = \frac{1}{\lambda} (rx, y) = \frac{1}{\lambda} (r, x, y)
\]

\[
(FIP_3)(x, y)(\lambda) = \frac{1}{\lambda} (x, y) = \frac{1}{\lambda} (\overline{y, x}) = \overline{\langle y, x \rangle (\lambda)}
\]

\[
(FIP_4)(x, x)(\lambda) = \frac{1}{\lambda} (x, x) = \langle x, x \rangle (\lambda) \geq 0
\]

Thus \( (H, \langle \cdot, \cdot \rangle) \) is fuzzy inner product space.

**Example 2.4:**

The space \( \mathbb{K}^n \) is an inner product space with inner product defined by

\[
\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n
\]

where \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \). Hence \( (\mathbb{K}^n, \langle \cdot, \cdot \rangle) \) is a fuzzy inner product space with fuzzy inner product defined by

\[
\langle x, y \rangle (\lambda) = \frac{1}{\lambda} (x, y)
\]

For every \( x = (x_1, x_2, \ldots, x_n) \), \( y = (y_1, y_2, \ldots, y_n) \)

where \( \lambda = \text{mtn}[\alpha, \beta] \) with \( \alpha, \beta \in (0, 1] \) by proposition 2.3.
Proposition 2.5:
If \((H, \langle \cdot, \cdot \rangle)\) is a fuzzy inner product space then \((H, \langle \cdot, \cdot \rangle)\) is an ordinary inner product space with,
\[
\langle x, y \rangle = \langle x, y \rangle(\lambda) \quad \text{for every } x, y \in H, \lambda \in (0,1]
\]
Proof:
Let
\[
x, y, z \in H \quad \text{and } r \in \mathbb{R}, \lambda \in (0,1)
\]
then
\[
(I_{P_1})(x + z, y) = \langle x + z, y \rangle(\lambda) = \langle x, y \rangle(\lambda) + \langle z, y \rangle(\lambda)
\]
\[
(I_{P_2})(x, y + z) = \langle x, y + z \rangle(\lambda) = \langle x, y \rangle(\lambda) + \langle x, z \rangle(\lambda)
\]
Thus \((H, \langle \cdot, \cdot \rangle)\) is an arbitrary inner product space.

Remark 2.6:
If \((H, \langle \cdot, \cdot \rangle)\) is a fuzzy inner product space then
(i)
\[
\langle rx + sy, z \rangle = r \langle x, z \rangle + s \langle y, z \rangle
\]
(ii)
\[
\langle x, ry + sz \rangle = r \langle x, y \rangle + s \langle x, z \rangle
\]
(iii)
\[
\langle x, ry + sz \rangle = r \langle x, y \rangle + s \langle x, z \rangle
\]
For all \(x, y, z \in H, \quad r, s \in \mathbb{R} \quad \text{where } a, b, c \in (0,1] \text{ with } \lambda = \min(a, b, c), r, s\)

Proof:

(i)
\[
\langle rx + sy, z \rangle = r \langle x, z \rangle + s \langle y, z \rangle \quad \text{by } (FIP_1)
\]

(ii)
\[
\langle x, ry + sz \rangle = r \langle x, y \rangle + s \langle x, z \rangle \quad \text{by } (FIP_2)
\]

(iii)
\[
\langle x, ry + sz \rangle = r \langle x, y \rangle + s \langle x, z \rangle \quad \text{by } (FIP_3)
\]

Theorem 2.7:[2]
If \(H, \langle \cdot, \cdot \rangle)\) is a fuzzy inner product space then
\[
|\langle x, y \rangle| \langle \lambda \rangle^2 \leq \langle x, x \rangle(\lambda) \cdot \langle y, y \rangle(\lambda)
\]
is called fuzzy Schwarz inequality,
where \(a, b \in (0,1) \text{ with } \lambda = \min(a, b)\)

Proof:
If \(x = 0\) then (1) is obvious. Put
\[
\beta = \langle x, y \rangle(\lambda) \quad \text{then}
\]
\[
0 \leq \langle yx + y^2, yx + y^2 \rangle(\lambda) = \langle yx, yx \rangle(\lambda) + \langle yx + y^2, y^2 \rangle(\lambda) = \langle yx, yx \rangle(\lambda) + \langle y + y^2, y^2 \rangle(\lambda)
\]
\[
|\lambda|^2 \langle x, x \rangle(\lambda) + |\lambda|^2 \langle y, x \rangle(\lambda) + |\lambda|^2 \langle x, y \rangle(\lambda) + |\lambda|^2 \langle y, y \rangle(\lambda)
\]
\[
0 \leq |\lambda|^2 \langle x, x \rangle(\lambda) + 2\Re\langle y, x \rangle(\lambda) + |\lambda|^2 \langle y, y \rangle(\lambda)
\]
Take \(\gamma = \frac{-\delta}{|\lambda|^2}\).
Now with this \(\gamma\) (2) becomes
\[
0 \leq |\lambda|^2 \langle x, x \rangle(\lambda) - \frac{|\delta|^2}{|\lambda|^2} \quad \text{or}
\]
\[
|\lambda|^2 \langle x, x \rangle(\lambda) \leq |\lambda|^2 \langle x, x \rangle(\lambda), \quad \text{or}
\]
\[
|\lambda|^2 \langle y, y \rangle(\lambda) \leq |\lambda|^2 \langle y, y \rangle(\lambda)
\]
Thus
\[
|\langle x, y \rangle(\lambda)|^2 \leq \langle x, x \rangle(\lambda) \cdot \langle y, y \rangle(\lambda)
\]

Theorem 2.8:[2]
Every fuzzy inner product space is a fuzzy normed space.

Proof:
Let \((H, \langle \cdot, \cdot \rangle)\) be a fuzzy inner product space. Define
\[
||x||(\lambda) = \langle x, x \rangle(\lambda)^{1/2}
\]
For each \(x, y, z \in H, \quad \text{where } a, b \in (0,1), \quad \text{let } x_a, y_b, z_c \in H\)
where \( \alpha, \beta, \gamma \in (0,1) \) with
\[
\lambda = \min\{\alpha, \beta, \gamma\} \text{ and } r \in \mathbb{R}
\]
\[
(FN_2)||x||_\lambda = \left(\|x\|^2_\lambda + \|y\|^2_\lambda + \|z\|^2_\lambda + \|w\|^2_\lambda\right)^{1/2}
\]
Thus
\[
Hence
\]
And there exist 
\[
\text{such that}
\]

### Theorem 2.9:

Let \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) be a fuzzy inner product space then the fuzzy norm
\[
||x||_\lambda = \left(\|x\|^2_\lambda + ||y||_\lambda^2\right)^{1/2}
\]
Satisfies the fuzzy parallelogram equality
\[
||x + y||^2_\lambda + ||x - y||^2_\lambda = 2(||x||^2_\lambda + ||y||^2_\lambda)
\]

\[
\text{for each } x, y \in \mathcal{H}
\]
where
\[
\alpha, \beta \in (0,1) \text{ with } \lambda = \min\{\alpha, \beta\}
\]
\[
||x + y||^2_\lambda = \langle x + y, x + y \rangle_\lambda
\]
\[
= \langle x, x \rangle_\lambda + \langle y, y \rangle_\lambda + \langle x, y \rangle_\lambda + \langle y, x \rangle_\lambda
\]
\[
= \langle x, x \rangle_\lambda + \langle y, y \rangle_\lambda
\]
Similarly, we have
\[
||x - y||^2_\lambda = ||x + y||^2_\lambda - 2||x||^2_\lambda
\]
Thus by adding these two equations we get
\[
||x + y||^2_\lambda + ||x - y||^2_\lambda = 2(||x||^2_\lambda + ||y||^2_\lambda)
\]

### Remark 2.10:

Not all fuzzy normed spaces are fuzzy inner product spaces, for example, the space \(\mathcal{C}[a, b]\) is a fuzzy normed space with fuzzy norm defined by
\[
||x||_\alpha = \sup_{\alpha \in [a,b]} \left(\frac{x}{\alpha}\right), \quad \beta = [a, b]
\]
But it is not a fuzzy inner product space.
We shows that the fuzzy norm cannot be obtained from a fuzzy inner product since this fuzzy norm does not satisfy the fuzzy parallelogram equality.
Indeed, if we take
\[
\langle x(t) = 1, \alpha \rangle \text{ and } \langle y(t) = \alpha, \beta \rangle \text{ where } \alpha, \beta \in (0,1)
\]
with \(\lambda = \min\{\alpha, \beta\}\)
We have
\[ \|x\| (\lambda) = \frac{1}{\lambda} \]
\[ \|y\| (\lambda) = \frac{1}{\lambda} \]
Hence
\[ x(t) + y(t) = 1 + \frac{\alpha}{b - a} \]
\[ x(t) - y(t) = 1 - \frac{\alpha}{b - a} \]
Thus
\[ \|x + y\| (\lambda) = \frac{2}{\lambda} \]
\[ \|x - y\| (\lambda) = \frac{2}{\lambda} \]
\[ \|x + y\|^2 (\lambda) + \|x - y\|^2 (\lambda) = \frac{5}{\lambda^2} \]

Definition 2.11:
A fuzzy element \( x_a \) of a fuzzy inner product space \( H \) is said to be orthogonal to a fuzzy element \( y_\beta \) in \( H \) if
\[ \langle x, y \rangle (\lambda) = 0 \]
where \( \alpha, \beta \in (0,1) \) with \( \lambda = \min(\alpha, \beta) \).
We also say that \( x_a \) and \( y_\beta \) are orthogonal, and we write \( x_a \perp y_\beta \).

Similarly for a fuzzy two sets \( A, B \) in \( H \) we write \( x_a \perp y_\beta \) if
\[ x_a \perp y_\beta \text{ for all } y_\beta \in B \text{ and } A \perp B \]
if \( x_a \perp y_\) for all \( x_a \) in \( A \) and \( y_\beta \) in \( B \).

3- Fuzzy Convergence, Fuzzy Cauchy Sequence, Fuzzy Complete

Definition 3.1:
A fuzzy sequence \( \{x_n \}_{n=1}^\infty \) in a fuzzy inner product space \( (H, \langle \cdot, \cdot \rangle) \) is said to be fuzzy convergent if there is a fuzzy vector \( x_\infty \) in \( H \) such that
\[ \lim_{n \to \infty} \|x_n - x_\infty\| (\lambda) = 0 \]
where \( \|x\| (\lambda) = \left[ \langle x, x \rangle (\lambda) \right]^{\frac{1}{2}} \).
and \( \alpha, \beta, \in (0,1) \) with \( \lambda = \min(\alpha, \beta) \).

With \( \lambda \in (0,1) \) such that
\[ \lambda = \inf_{\lambda \in (0,1)} \sup_{n \in \mathbb{N}} \|x_n - x_\infty\| (\lambda) \]

Definition 3.2:
A fuzzy sequence \( \{x_n \}_{n=1}^\infty \) in a fuzzy inner product space \( (H, \langle \cdot, \cdot \rangle) \) is said to be fuzzy Cauchy if for every \( \varepsilon > 0 \), there is \( M > 0 \) such that \( \|x_m - x_n\| < \varepsilon \)
for every \( m, n > M \) where
\[ \|x\| (\lambda) = \left[ \langle x, x \rangle (\lambda) \right]^{\frac{1}{2}} \]
with
\[ \lambda = \min(\alpha, \beta, \in (0,1) \text{ such that } \lambda \in (0,1) \text{ and } \lambda \in \mathbb{N} \]

Definition 3.3:
A fuzzy inner product space \( (H, \langle \cdot, \cdot \rangle) \) is said to be fuzzy complete if every fuzzy Cauchy sequence in \( H \) converges, that is, has a fuzzy limit which is a fuzzy vector of \( H \).

Proposition 3.4:
Let \( \left\{ \langle x_n, y_n \rangle \right\} \) be a fuzzy inner product space, if \( \left\{ x_n \right\} \) fuzzy sequence converges to \( x_\infty \) and \( \left\{ y_n \right\} \) fuzzy sequence converges to \( y_\infty \), then
\[ \left\{ \langle x_n, y_n \rangle \right\} \text{ fuzzy sequence converges to } \langle x_\infty, y_\infty \rangle \]
where \( \lambda = \min(\alpha, \beta, \in (0,1) \text{ such that } \lambda \in (0,1) \text{ and } \lambda \in \mathbb{N} \)

Proof:
Subtracting and adding a term, using triangle inequality for numbers and finally, the fuzzy Schwarz in equality,
We obtain
\[ \|x_n - x_\| \leq \|x_n - y_\| + \|y_\| \]
\[ \|x_n - x_\| \leq ||x_n - x_\| \|y_\| \|

Since \( \left\{ y_n \right\} \) converges to zero and \( \left\{ x_n \right\} \) converges to \( x_\infty \) as \( n \to \infty \).
Definition 3.5:
A fuzzy inner product space \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) is said to be a fuzzy Hilbert space if it is fuzzy complete with respect to the fuzzy normed \(\|x\|_\alpha = \left[\langle x, x \rangle_\alpha \right]^\frac{1}{2}\) where \(x_\alpha \in \mathcal{H}\)

Theorem 3.6:
If \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) is an ordinary Hilbert space, then \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) is a fuzzy Hilbert space where \(\langle x, y \rangle_\lambda = \frac{1}{\lambda} \langle x, y \rangle\), for every \(x_\alpha, y_\beta \in \mathcal{H}\) with \(\lambda = \min[\alpha, \beta \in (0,1)]\).

Proof:
\((\mathcal{H}, \langle \cdot, \cdot \rangle)\) is a fuzzy inner product space by proposition 2.3 since \((\mathcal{H}, \| \cdot \|_\lambda)\) is complete normed space where \(\|x\|_\lambda = \left[\langle x, x \rangle_\lambda \right]^\frac{1}{2}\) so \((\mathcal{H}, \| \cdot \|_\lambda)\) is a fuzzy complete normed space where:

\[\|x\|_\lambda = \frac{1}{\lambda} \|x\|\] for all \(x_\alpha \in \mathcal{H}\).

Thus \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) is a fuzzy Hilbert space

Theorem 3.7:
If \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) is a fuzzy Hilbert space then \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) is an ordinary Hilbert space where \(\langle x, y \rangle = \langle x, y \rangle_\lambda\) where \(\lambda \in (0,1]\).

Proof:
First \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) is an ordinary inner product space by proposition 2.6, since \((\mathcal{H}, \| \cdot \|_\lambda)\) is fuzzy complete normed space where \(\|x\|_\lambda = \left[\langle x, x \rangle_\lambda \right]^\frac{1}{2}\) so \((\mathcal{H}, \| \cdot \|_\lambda)\) is complete normed space where \(\|x\| = \|x\|_\lambda\) where \(\alpha \in (0,1]\).

Thus \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) is an ordinary Hilbert space.

Example 3.8:
The space \(\mathbb{C}^n\) is a Hilbert space with inner product defined by

\[\langle x, y \rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n\]

Where

\[x = (x_1, x_2, \ldots, x_n), \ y = (y_1, y_2, \ldots, y_n)\]

See [1]
Hence \((\mathbb{C}^n, \langle \cdot, \cdot \rangle)\) is a fuzzy Hilbert space with fuzzy inner product defined by

\[\langle x, y \rangle_\lambda = \frac{1}{\lambda} \langle x, y \rangle\]

Where \(\lambda = \min[\alpha, \beta \in (0,1)]\) by theorem 3.6

Theorem 3.9: [3]
Let \((\mathcal{X}, \| \cdot \|_\lambda)\) be a fuzzy normed space. Then there is a fuzzy Banach space \(\mathcal{Y}\) and a fuzzy isometry \(I\) from \(\mathcal{X}\) onto a fuzzy subspace \(\mathcal{W}\) of \(\mathcal{Y}\) which is fuzzy dense in \(\mathcal{Y}\). The space \(\mathcal{Y}\) is unique except for fuzzy isometries.

Definition 3.10:
An isomorphism \(T\) of a fuzzy inner product space \(\mathcal{H}\) onto a fuzzy inner product space \(\mathcal{Y}\) over the same field is an injective linear operator \(T: \mathcal{H} \to \mathcal{Y}\) which preserves the fuzzy inner product that is for all \(x_\alpha, y_\beta \in \mathcal{H}\) with

\[\lambda = \min[\alpha, \beta \in (0,1)]\]

\[\langle T(x), T(y) \rangle_\lambda = \langle x, y \rangle_\lambda\]

Thus \(\mathcal{H}\) is an ordinary Hilbert space.

Note that the bijectivity and linearity guarantees that \(T\) is a vector space isomorphism of \(\mathcal{H}\) onto \(\mathcal{Y}\), so that \(T\) preserves the whole structure of fuzzy inner product. \(T\) is also an isometry of \(\mathcal{H}\) onto \(\mathcal{Y}\) because fuzzy distances in \(\mathcal{H}\) and \(\mathcal{Y}\) are determined by the fuzzy norms defined by the fuzzy inner products on \(\mathcal{H}\) and \(\mathcal{Y}\).

The theorem about the fuzzy completion of a fuzzy inner product can now stated as follows:

...
Theorem 3.11:
For any fuzzy inner product space $H$ there exists a fuzzy Hilbert space $Y$ an isomorphism $T$ from $H$ onto a fuzzy dense fuzzy subspace $\tilde{Y}$ of $Y$. the space $Y$ is unique except for isomorphisms.

Proof:
By theorem 3.10 there exists a fuzzy Banach space $Y$ and an isometry $T$ from $H$ onto a fuzzy subspace $\tilde{Y}$ of $Y$ which is fuzzy dense in $H$, for reasons of continuity, under such an isometry, sums and scalar multiples of elements in $H$ and $\tilde{Y}$ correspond to each other, so that $T$ is even an isomorphism of $H$ onto $\tilde{Y}$, both regarded as fuzzy normed spaces. Proposition 3.4 shows that we can define a fuzzy inner product on $Y$ by setting:

$$\langle [x], [y] \rangle (\lambda) = \lim_{\alpha \to \infty} \langle x_\alpha, y_\alpha \rangle (\lambda)$$

The notation being as in theorem 3.10, that is $\{[x_\alpha, \alpha]\}$ and $\{[y_\alpha, \alpha]\}$ are representative of $[x] \in Y$ and $[y] \in Y$ respectively we see that $T$ is an isomorphism of $H$ onto $\tilde{Y}$, both regarded as fuzzy inner product spaces.

Theorem 3.10 also guarantees that $Y$ is unique except for isometries that is, two completions $Y$ and $Z$ of $H$ are related by an isometry $A: Y \to Z$. Reasoning as in the case of $T$, we conclude that $A$ must be an isomorphism of the fuzzy Hilbert space $Y$ onto the fuzzy Hilbert space $Z$.

Lemma 3.12:
Let $(H, \langle \cdot, \cdot \rangle)$ be a fuzzy inner product space. If

$$x \perp \gamma \text{ then } \|x + y\|^2(\lambda) = \|x\|^2(\lambda) + \|y\|^2(\lambda)$$

where $\lambda = \min \{\alpha, \beta \in (0,1]\}$.

Proof:
$$\|x + y\|^2(\lambda) = \langle x + y, x + y \rangle (\lambda) = \langle x, x \rangle (\lambda) + \langle y, y \rangle (\lambda) + 2 \langle x, y \rangle (\lambda)$$
$$= (\langle x_\alpha, x_\alpha \rangle (\lambda) + \langle y_\alpha, y_\alpha \rangle (\lambda) + 2 \langle x_\alpha, y_\alpha \rangle (\lambda)$$
$$= (\langle x, x \rangle (\lambda) + \langle y, y \rangle (\lambda) + 2 \langle x, y \rangle (\lambda)$$

Then
$$\|x + y\|^2(\lambda) = \|x\|^2(\lambda) + \|y\|^2(\lambda)$$
Since $\langle x, y \rangle (\lambda) = 0$.

References