Least Squares Method For Solving Integral Equations With Multiple Time Lags

Dr.Suha N. Shehab*, Hayat Adel Ali* & Hala Mohammed Yaseen*
Received on:26/10/2009
Accepted on:11/3/2010

Abstract
The main purpose of this work is to propose an approximate method to solve integral equation with multiple time lags (IEMTL) namely least squares method with aid of Chebyshev polynomials of (first, second, third, and fourth) kinds. Example is given as an application of least squares method with aid of four kinds of Chebyshev polynomials.

Keywords: Integral Equation, time lag, Least squares method, Chebychev polynomials

1. Introduction
The most recent kind of equation that worth studying is the delay integral equation. These equations have many applications like: a model to explain the observed periodic out breaks of certain infection diseases [12]. Another application is the electromagnetic inverse scattering problem in a medium with discontinuous change in conductivity [3]. It will be necessary to give simple information about the integral equation with multiple time lags.

2. Integral Equations with Multiple Time Lags [4, 5, 9]
The significance of these equations lies in ability to describe processes with retarded (delay) time which may appear in the unknown function $f(t)$ involved in the integrand or may appear in the unknown function $f(t)$ in the left hand side of the equation or may appear in one of the limits of the integrations.
The integral equation with multiple time lags (which have two lags $\tau_1$ and $\tau_2$ such that $\tau_1, \tau_2 \in R$, $\tau_1$ and $\tau_2 > 0$) having the following cases:-

*Applied Sciences Department, University of Technology/Baghdad

https://doi.org/10.30684/etj.28.10.2
2412-0758/University of Technology-Iraq, Baghdad, Iraq
This is an open access article under the CC BY 4.0 license http://creativecommons.org/licenses/by/4.0
• If $\tau_1$ appears in the unknown function $f(t)$ inside the integral sign and $\tau_2$ appears in the unknown function $f(t)$ outside the integral sign such that

$$h(t)f(t-\tau_2) = g(t) + \int_a^{b(t)} k(t,y)f(y-\tau_1)dy$$

...(1)

• If $\tau_1$ appears in the unknown function $f(t)$ inside the integral sign and $\tau_2$ appears in one of the limits of integration

$$h(t)f(t) = g(t) + \int_a^{\tau_2} k(t,y)f(y-\tau_1)dy$$

or

$$h(t)f(t) = g(t) + \int_{\tau_2}^{b(t)} k(t,y)f(y-\tau_1)dy$$

...(2)

• If $\tau_1$ appears in the unknown function $f(t)$ outside the integral sign and $\tau_2$ appears in one of the limits of integration:

$$h(t)f(t-\tau_1) = g(t) + \int_a^{\tau_2} k(t,y)f(y)dy$$

...(4)

or

$$h(t)f(t-\tau_1) = g(t) + \int_{\tau_2}^{b(t)} k(t,y)f(y)dy$$

...(5)

Remarks [1, 5, 6]:
1. If $h(t) = 0$ these equations are called integral equation with multiple time lags of the first kind.
2. If $h(t) = 1$ the equations are called integral equation with multiple time lags of the second kind.
3. If $g(t) = 0$ these equations are called homogeneous integral equations with multiple time lags otherwise if $g(t) \neq 0$ the equation are called nonhomogenous integral equation with multiple time lags.
4. If $b(t) = t$ the integral equation is called Volterra integral equation with time lag otherwise when $(b(t) = b, b$ is a constant) it is called Fredholm integral equation with time lag.

3. Chebyshev Polynomials [8, 10]:-

Chebyshev polynomials are of great importance in many area of mathematics particularly approximation theory. Numerous articles and books have been written about this topic. There are several kinds of Chebyshev polynomials. In particular we shall introduce the first, second, third and fourth kind of Chebyshev polynomials.

Some books and many articles use the expression Chebyshev polynomial to refer exclusively to the Chebyshev polynomial $T_n(x)$ of the first kind. Indeed this is by far the most important of the Chebyshev polynomials and when no other qualification is given.

Clearly some definition of Chebyshev polynomials is needed right away so a choice of definitions. However, what gives the various polynomials their power and relevance is their close relationship with the trigonometric functions 'cosine' and 'sine'. It must be aware of the power of these functions and of their appearance in the description of all kinds of natural phenomena, and this must surely be
the key to the versatility of Chebyshev polynomials. It will be necessary to make some primary definitions of these trigonometric relationships.

3.1 The First-Kind Chebyshev Polynomials \( T_n(x) \) \[2\]

The Chebyshev polynomial \( T_n(x) \) of the first kind is a polynomial in \( x \) of degree \( n \), and range of the variable \( x \) in the interval \([-1,1]\) then it will be defined by the relation.

\[
T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \quad n = 2,3,...
\]  

which may readily show that.

\[
T_0(x) = 1 \quad \text{ ..........(6)}
\]

\[
T_1(x) = x \quad \text{ ..........(8)}
\]

3.2 The Second-Kind Chebyshev Polynomial \( U_n(x) \) \[8, 11\]

The Chebyshev polynomial \( U_n(x) \) of the second kind is a polynomial of degree \( n \) in \( x \) and range same as for \( T_n(x) \) defined by the recurrence

\[
U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x) \quad n = 2,3,...
\]

\[
\text{with} \quad U_0(x) = 1, U_1(x) = 2x
\]

so we deduce that

\[
U_0(x) = 1, \quad U_1(x) = 2x, \quad U_2(x) = 4x^2 - 1, \quad U_3(x) = 8x^3 - 4x
\]

3.3 The Third and Fourth-Kind Chebyshev Polynomial \( V_n(x) \) and \( W_n(x) \) (the airfoil polynomial) \[8\]

Two other families of polynomials \( V_n(x) \) and \( W_n(x) \) may be constructed, which are related to \( T_n(x) \) and \( U_n(x) \), but which have trigonometric definitions involving the half angle \( \theta/2 \) (where \( x = \cos \theta \) as before). These polynomials are same time referred to as the 'airfoil polynomials', which is rather a appropriately named third and fourth-kind Chebyshev polynomials.

3.3.1 The Chebyshev polynomials \( V_n(x) \) and \( W_n(x) \) of the third and fourth kinds are polynomials of degree \( n \) in \( x \):

These polynomials too may be efficiently generated by the use of a recurrence relation.

\[
V_n(x) = 2x V_{n-1}(x) - V_{n-2}(x) \quad \text{ ..........(11)}
\]

\[
\text{with} \quad V_0(x) = 1, \quad V_1(x) = 2x - 1
\]

\[
W_n(x) = 2x W_{n-1}(x) - W_{n-2}(x) \quad \text{ ..........(12)}
\]

\[
\text{with} \quad W_0(x) = 1, \quad W_1(x) = 2x + 1
\]

where \( n = 2,3,4,... \)

we may readily show that

\[
V_2(x) = 4x^2 - 4x, \quad V_3(x) = 8x^3 - 4x - 1
\]

Thus \( V_n(x) \) and \( W_n(x) \) share precisely the same recurrence relation as \( T_n(x) \), \( U_n(x) \) and their generation differs only in the prescriptions of the initial condition \( n=1 \).

4. Chebyshev Polynomials for the General Range \([a,b]\) \[8\]

For more generally Chebyshev polynomials may define appropriate to any given finite range \([a,b]\) of \( x \), by making this range correspond to the range \([-1,1]\)
of a new variable $t$ under the linear transformation
\[ t = \frac{2x - (a + b)}{b - a} \]

### 4.1 Shifted Chebyshev Polynomials [8]:-

The shifted polynomials $T_n^*(x), U_n^*(x), V_n^*(x), W_n^*(x)$ since the range $[0,1]$ is quite often more convenient to use than the range $[-1,1]$ we sometimes map the independent variable $x$ in $[0,1]$ to the variable $t$ in $[-1,1]$ by the transformation
\[ t = 2x - 1 \quad \text{or} \quad x = \frac{t + 1}{2} \]
and this lead to a shifted polynomial (of the first kind) $T_n^*(x)$ of degree $n$ in $x$ on $[0,1]$ given by
\[ T_n^*(x) = T_n(t) = T_n(2x - 1) \quad \ldots(13) \]
The recurrence relation for $T_n^*$ may deduce in the form
\[ T_{n+1}^* = 2(2x - 1)T_n^* - T_{n-1}^* \quad \ldots(14) \]
with the initial conditions
\[ T_0^* = 1, \quad T_1^* = 2x - 1 \quad \ldots(15) \]

shifted polynomials $U_n^*(x), V_n^*(x), W_n^*(x)$ of the second, third and fourth kinds may be defined in precisely analogous ways:
\[ U_n^*(x) = U_n(2x - 1), \]
\[ V_n^*(x) = V_n(2x - 1), \]
\[ W_n^*(x) = W_n(2x - 1) \quad \ldots(16) \]

### 5. The Least Squares Method for Solving IEMTL [3, 7]:-

This method is one of the approximate methods used to solve the integral equations without time lag.

This method can be used to find an approximate solution for the integral equations with multiple time lags. To do this, consider the linear Fredholm integral equation with multiple time lags:
\[ f(t - \tau_j) = g(t) + \int_{\tau_j}^{t} k(t, y) f(y - \tau_j) \, dy \quad \ldots(17) \]

This method is based on approximating the unknown function $f$ as:
\[ f(t) \approx \sum_{i=0}^{n} c_i Q_i(t) \]

where $Q_i(t)$ is chosen to be one of the four kind of chebyshev polynomials ($T_i(t), U_i(t), V_i(t)$ or $W_i(t)$) by substituting these solutions into equ. (17). One can obtain
\[ \sum_{i=0}^{n} \left[ Q_i(t) - g(t) - \int_{\tau_j}^{t} k(t, y) Q_i(y - \tau_j) \, dy \right]^2 dt = R(t, c_0, c_1, \ldots, c_n) \ldots(18) \]

let
\[ M(c_0, c_1, \ldots, c_n) = \int_{a}^{b} \left[ R(t, c_0, c_1, \ldots, c_n) \right]^2 w(t) \, dt \]

Where $w(t)$ is any positive function defined on the interval $[a, b]$. It is usually called the weight function. In this work we take $w(t) = 1$ for simplicity.

Thus
\[ M(c_0, c_1, \ldots, c_n) = \int_{a}^{b} \left[ \sum_{i=0}^{n} \left[ Q_i(t) - g(t) - \int_{\tau_j}^{t} k(t, y) Q_i(y - \tau_j) \, dy \right]^2 \right] dt \]

So, finding the values of $c_i$ $i=0, 1, 2, \ldots, n$ which minimize $M$ is
equivalent to finding the best approximation for the solution of the integral equation given by eq.(17). The minimum value of $M$ is obtained by setting:

$$\frac{\partial M}{\partial c_j} = 0, \ j=0,1,2,\ldots,n$$

\[ \int \sum_{i=0}^{n} c_i Q_i(t - \tau_i) - \int k(t,y) \sum_{i=0}^{n} c_i Q_i(y - \tau_i) dy = R(t,c_i) \]

thus

\[ \int R(t,c_0,c_1,\ldots,c_n)\left\{ Q_i(t - \tau_i) - \int k(t,y)Q_i(y - \tau_i)dy\right\} dt = 0 \]

By the above equation for $j=0,1,\ldots,n$ one can obtain a system of $n+1$ linear equations with $n+1$ unknown $c_i$'s. This system can be formed by using matrices form as follows:

$$A = \left[ \begin{array}{cccc}
\int_{a}^{b} R(t,c_0)h_0 dt & \int_{a}^{b} R(t,c_0)h_1 dt & \cdots & \int_{a}^{b} R(t,c_0)h_n dt \\
\int_{a}^{b} R(t,c_1)h_0 dt & \int_{a}^{b} R(t,c_1)h_1 dt & \cdots & \int_{a}^{b} R(t,c_1)h_n dt \\
\vdots & \vdots & \ddots & \vdots \\
\int_{a}^{b} R(t,c_n)h_0 dt & \int_{a}^{b} R(t,c_n)h_1 dt & \cdots & \int_{a}^{b} R(t,c_n)h_n dt
\end{array} \right]$$

$$, \quad b = \left[ \begin{array}{c}
\int_{a}^{b} g(t)h_0 dt \\
\cdots \\
\int_{a}^{b} g(t)h_n dt
\end{array} \right]$$

where

$$h_j = Q_j(t - \tau_j) - \int_{a}^{b} k(t,y)Q_j(y - \tau_j) dy \quad j=0,1,2,\ldots,n$$

By solving the above system using Gauss elimination method to get an approximation solution to the eq.(17).

6. Numerical Example:-

Consider the Volterra integral equation with multiple time lags $f(t - \tau_i) = (t^2 - 1)^i \quad i \in [1,2]$.

Where $\tau_1 = 1$ and $\tau_2 = 1$ with the exact solution $f(t) = (t^2 + 1)$.

Assume that the approximate solution is

1. $f(t) \approx f_2(t) = \sum_{i=0}^{2} c_i T_i(2t - 3)$
2. $f(t) \approx f_2(t) = \sum_{i=0}^{2} c_i U_i(2t - 3)$
3. $f(t) \approx f_2(t) = \sum_{i=0}^{2} c_i V_i(2t - 3)$
4. $f(t) \approx f_2(t) = \sum_{i=0}^{2} c_i W_i(2t - 3)$

The least squares method with shifted Chebychev $(1^{st}, 2^{nd}, 3^{rd}, 4^{th})$ approximation used to solve this problem. Their results are obtained by using matlab program. The following table(1) shows a comparison between the exact and the approximated results by absolute error (Abs.E) and least square error (L.S.E) between the methods.
6. Conclusions
1. The results obtained by using least squares method are very accurate in a comparison with the exact solution of worked examples.
2. If we were asked for a pecking order of these four chebyshev polynomials
   \( T_n(x), U_n(x), V_n(x), W_n(x) \) then we say:
   (a) \( T_n(x) \) is clearly the most important and \( T_n(x) \) generally leads to simplest formulae, whereas results for the other polynomials may involve slight complications.
   (b) All four polynomials have their role. For example \( U_n(x) \) is useful in numerical integration, while \( V_n(x) \) and \( W_n(x) \) can be useful in situations in which singularities occur at one end points (+1 or −1) but not at the other.
3. The results and the required duration of the least squares method with the aid of four kinds of chebyshev polynomials (1st, 2nd, 3rd, 4th) declare that the least square method with aid of (3rd, 4th) kinds gives more accurate results than the others.
4. A disadvantage of the least squares method with aid of four kinds of chebyshev polynomials is dependence on a free parameter \( n \) which gives the smallest least square error.

References

Table (1)

<table>
<thead>
<tr>
<th>t</th>
<th>Exact</th>
<th>Least Square</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>first Cheby</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1.9701</td>
</tr>
<tr>
<td>1.1</td>
<td>2.21</td>
<td>2.1819</td>
</tr>
<tr>
<td>1.2</td>
<td>2.44</td>
<td>2.41399</td>
</tr>
<tr>
<td>1.3</td>
<td>2.69</td>
<td>2.667</td>
</tr>
<tr>
<td>1.4</td>
<td>2.96</td>
<td>2.9407</td>
</tr>
<tr>
<td>1.5</td>
<td>3.25</td>
<td>3.2458</td>
</tr>
<tr>
<td>1.6</td>
<td>3.56</td>
<td>3.5509</td>
</tr>
<tr>
<td>1.7</td>
<td>3.89</td>
<td>3.8873</td>
</tr>
<tr>
<td>1.8</td>
<td>4.24</td>
<td>4.2446</td>
</tr>
<tr>
<td>1.9</td>
<td>4.61</td>
<td>4.6227</td>
</tr>
<tr>
<td>2</td>
<td>5.00</td>
<td>5.0217</td>
</tr>
<tr>
<td>L.E</td>
<td></td>
<td>4.023e-3</td>
</tr>
</tbody>
</table>

Figure (1) shows the comparison between these results.