Approximate Solution For Linear Fredhlom Integro-Differential Equation By Using Bernstein Polynomials Method

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Abstract

In this paper, Bernstein polynomials method is used to find an approximate solution for Fredholm integro-Differential equation of the second kind. These polynomials are incredibly useful mathematical tools, because they are simply defined, can be calculated quickly on computer systems and represent a tremendous variety of functions. They can be differentiated and integrated easily.

Keywords: Integro-Differential Equation, and Bernstein Polynomials.

حل تقريبي لمعادلة فريدهولوم التفاضلية الخطية باستخدام متعددة حدود برنشتن

الخلاصة

في هذا البحث استعملت طريقة متعددة حدود برنشتن لإيجاد الحل التقريبي لمعادلة فريدهولوم التفاضلية الخطية من النوع الثاني. وأن متعددات الحدود تستعمل بشكل كبير في الطرق الرياضية وذلك لبساطة تعريفها، والحل بهذه الطريقة يتقارب بسرعة وبخطوات قليلة. كما يمكن تفاضلها وتكاملها بسهولة.

1. Introduction

Several numerical methods for approximating Fredholm integro-differential equation are know.

In this paper, we introduce approximate method for solving linear Fredholm integro-differential equation by using Bernstein polynomials method. Hence, we begin by giving a general introduction to integral equations.

An integral equation is generally defined as an equation which involves the integral of an unknown function.

A linear integral equation is an integral equation which involves a linear expression of the unknown function.

The Fredholm integral equations are integral equations divided into two group, referred to as Fredholm integral equations of the first or the second kind.

They have the following general expression[1][2][6]:

$$y(x) = f(x) + \int_{\Omega} k(x,t)\, y(t)\, dt \quad \ldots(1)$$

where k(x,t) and f(x) are known function. k(x,t) is called the kernel of integral equation. Y(x) is the function to be determined. When Ω is finite interval $[a,b] \subseteq R$ in this case, the fredholm integral equations of first and second kind will respectively have the following expressions:

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$$f(x)=\int_a^b k(x,t)y(t)dt \quad x\in[a,b] \qquad \ldots(2)$$

$$y(x)=f(x)+\int_a^b k(x,t)y(t)dt$$

$$x\in[a,b] \qquad \ldots(3)$$

There for the fredholm integro-differential equation by[5][8]:

$$y'(x)=f(x)+\int_a^b k(x,t)y(t)dt$$

$$y(a)=y_a$$

$$a\le x\le b \qquad \ldots(4)$$

In this paper, we introduce approximation method to solve the linear fredholm integro-differential equation of the second kind by using Benstein polynomials method.

2. Bernstein Polynomials Method

Polynomials are incredibly useful mathematical tools as they are simply defined, can be calculated quickly on computer systems and represent a tremendous variety of functions.

The Bernstein polynomials of degree n are defined by [3], [4].

$$B_i^n(t)=\binom{n}{i}t^i(1-t)^{n-i}$$

$$\text{for } i=0,1,2,\ldots,n \qquad \ldots(5)$$

where

$\binom{n}{i}=\dfrac{n!}{i!(n-i)!}$, (n) is the degree of polynomials, (i) is the index of polynomials and (t) is the variable.

The exponents on the (t) term increase by one as (i) increases, and the exponents on the (1-t) term decrease by one as (i) increases.

The Bernstein polynomial of degree (n) can be defined by blending together two Bernstein polynomials of degree (n-1). That is, the n^{th} degree Bernstein polynomial can be written as, [4].

$$B_k^n(t)=(1-t)B_k^{n-1}(t)+tB_{k-1}^{n-1}(t) \qquad \ldots(6)$$

Bernstein polynomials of degree (n) can be written in terms of the power basis. This can be directly calculated using the equation (5) and the binomial theorem as follows, [4].

$$B_k^n(t)=\binom{n}{k}t^k(1-t)^{n-k}=\sum_{i=k}^{n}(-1)^{i-k}\binom{n}{i}\binom{i}{k}t^i$$

Where the binomial theorem is used to Expand $(1-t)^{n-k}$.

The derivatives of the nth degree Bernstein polynomials are polynomials of degree (n-1)

$$\frac{d}{dt}B_k^n(t)=\frac{d}{dt}\binom{n}{k}t^k(1-t)^{n-k}$$

$$=n\left(B_{k-1}^{n-1}(t)-B_k^{n-1}\right) \qquad 0\le k\le n \qquad \ldots(7)$$

3. A Matrix Representation for Bernstein Polynomials

In many applications, a matrix formulation for the Bernstein polynomials is useful. These are straight forward to develop if only looking at a linear combination in terms of dot products. Given a polynomial written as a linear combination of the Bernstein basis functions [3][4].

$$B(t)=c_0B_0^n(t)+c_1B_1^n(t)+c_2B_2^n(t)+\ldots+c_nB_n^n(t) \qquad \ldots(8)$$

It is easy to write this as a dot product of two vectors

$$B(t)=\left[B_0^n(t)\ \ B_1^n(t)\ \ B_2^n(t)\ \ldots B_n^n(t)\right]\begin{bmatrix}c_0\\c_1\\\vdots\\c_n\end{bmatrix}$$

…(9)

which can be converted to the following form:

$$B(t)=\left[1\ t\ t^2\ \ldots t^n\right]\begin{bmatrix}b_{00}&0&0&\ldots&0\\b_{10}&b_{11}&0&\ldots&0\\b_{20}&b_{21}&b_{22}&\ldots&0\\\vdots&\vdots&\vdots&&\vdots\\b_{n0}&b_{n1}&b_{n2}&\ldots&b_{nn}\end{bmatrix}\begin{bmatrix}c_0\\c_1\\c_2\\\vdots\\c_n\end{bmatrix}$$

… (10)

where b_{nn} are the coefficients of the power basis that are used to determine the respective Bernstein polynomials, we note that the matrix in this case lower triangular.

The matrix of derivatives of Brenstein polynomials

$$B'(t)=\left[0\ 1\ 2t\ \ldots nt^{n-1}\right]\begin{bmatrix}b_{00}&0&0&\ldots&0\\b_{10}&b_{11}&0&\ldots&0\\b_{20}&b_{21}&b_{22}&\ldots&0\\\vdots&\vdots&\vdots&&\vdots\\b_{n0}&b_{n1}&b_{n2}&\ldots&b_{nn}\end{bmatrix}\begin{bmatrix}c_0\\c_1\\c_2\\\vdots\\c_n\end{bmatrix}$$

… (10a)

4. Solution of Fredholm Integro-differential Equation with Bernstein Polynomials

In this section Bernstein polynomials to find the approximate solution for Fredholm integro-differential equation, will be introduced [4][5][7].

Let us reconsider the Fredholm integro-differential equation of the second kind in equation (4).

$$y'(x)=f(x)+\int_a^b k(x,t)y(t)dt \qquad x\in[a,b]$$

… (11)

And

Let

$$y(t)=B(t)=\left[B_0^n(t)\ \ B_1^n(t)\ \ B_2^n(t)\ldots B_n^n(t)\right]\begin{bmatrix}c_0\\c_1\\\vdots\\c_n\end{bmatrix}$$

by using equation (9) $y'(t)=n\left(B_{k-1}^{n-1}(t)-B_k^{n-1}\right)$ by using equation (7)

A applying the Bernstein polynomials method for equation (11), we get the following formula.

$$\left[0\ 1\ \ B_{1-1}^{n-1}(t)\ldots B_n^{n-1}(t)\right]\begin{bmatrix}c_0\\c_1\\\vdots\\c_n\end{bmatrix}=$$

$$f(x)+\int_a^b k(x,t)[B_0^n(t)\ \ B_1^n(t)\ldots B_n^n(t)]\begin{bmatrix}c_0\\c_1\\\vdots\\c_n\end{bmatrix}dt$$

…(12)

by using equations (10) and ()10a), which can be converted to the following form:

$$[0\ 1\ 2t \cdots nt^{n-1}]\begin{bmatrix} b_{00} & 0 & \cdots & 0 \\ b_{10} & b_{11} & \cdots & 0 \\ b_{20} & b_{21} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ b_{n0} & b_{n1} & & b_{nn} \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} =$$

$$f(x)+\int_a^b k(x,t)[1\ t \cdots t^n]\begin{bmatrix} b_{00} & 0 & \cdots & 0 \\ b_{10} & b_{11} & \cdots & 0 \\ b_{20} & b_{21} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ b_{n0} & b_{n1} & & b_{nn} \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} dt \quad \ldots (13)$$

now to find all integration in equation(13).

Then in order to determine $c_0, c_1, \ldots, c_n$, we need n equations;

Now Choice $x_i, i = 1,2,3,\ldots n$ in the interval [a, b], which gives (n) equations.

Solve the (n) equations by Gauss elimination to find the values $c_0, c_1, \ldots, c_n$.

The following algorithm summarizes the steps for finding the approximate solution for the second kind of linear Fredholm integro-differential equation .

5. Algorithm (BPM)

Input: $(f(t), k(t,s), y(s), a, b)$,

Output: polynomials of degree n

Step1:

Choice n the degree of Bernstein polynomials

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i} \quad \text{for}$$

$i = 0,1,2,\ldots,n$

Step2:

Put the Bernstein polynomials in linear Fredholm integro differential equation of second kind.

$$y'(t) = [0\ 1\ 2t\ \cdots\ nt^{n-1}]\begin{bmatrix} b_{00} & 0 & 0 & \cdots & 0 \\ b_{10} & b_{11} & 0 & \cdots & 0 \\ b_{20} & b_{21} & b_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n0} & b_{n1} & b_{n2} & & b_{nn} \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$= f(x) + \int_a^b k(x,t)[B_0^n(t)\ B_1^n(t) \cdots B_n^n(t)]\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} dt$$

Step3:

Compute

$$\int_a^b k(x,t)[1\ t \cdots t^n]\begin{bmatrix} b_{00} & 0 & 0 & \cdots & 0 \\ b_{10} & b_{11} & 0 & \cdots & 0 \\ b_{20} & b_{21} & b_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n0} & b_{n1} & b_{n2} & & b_{nn} \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} dt$$

Compute

$$[0\ 1\ 2t \cdots nt^{n}]\begin{bmatrix} b_{00} & 0 & 0 & \cdots & 0 \\ b_{10} & b_{11} & 0 & \cdots & 0 \\ b_{20} & b_{21} & b_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n0} & b_{n1} & b_{n2} & & b_{nn} \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Step4:

Compute $c_0, c_1, \cdots, c_n$, where $x_i, i = 1,2,3,\cdots,n$, $x_i \in [a,b]$

End:

Example:
Consider the following linear Fredholm integro-differential equation of the second kind:

$$y'(x) = xe^{x} + e^{x} - x + \int_0^1 xy(t)dt$$

With initial condition $y(0)=0$, and with the exact solution $y(x)=xe^{x}$
Now to derive the solution by using the Bernstein polynomials method, we can use the following scheme:
When Bernstein polynomials algorithm is applied both sides in example. And choice the degree of Bernstein polynomials n=2, we get:

$$-2a_0(1-x)+2a_1(1-x)-2c_1x+2c_2x$$

$$=xe^{x}+e^{x}-x+\int_0^1 x\left[c_0(1-t)^2+2c_1t(1-t)+c_2t^2\right]dt$$

Next

$$-2c_0(1-x)+2c_1(1-x)-2c_1x)+2c_2x$$

$$=xe^{x}+e^{x}-x+\left(2c_0\int_0^1 x(1-t)^2dt+2c_1\int_0^1 xt(1-t)dt+c_2\int_0^1 xt^2dt\right)$$

And after performing the integration.

$$-2c_0(1-x)+2c_1-4c_1x+2c_2x$$

$$=xe^{x}+e^{x}-x+\left[\frac{c_0}{3}+c_1+\frac{c_2}{3}\right]x-\frac{2}{3}c_1$$

$$(-2+\frac{2}{3}x)c_0+(\frac{8}{3}-5x)c_1+(\frac{5}{3}x)c_2=xe^{x}+e^{x}-x$$

Then in order to determine c_0, c_1 *and* c_2, we need three equation;
Now Choice $x_i, i=1,2$ in the interval [0, 1], with substation in the initial condition in the equation $y'(x)=-2a_0(1-x)+2a_1-4c_1x+2c_2x$ which gives three equations.

$$c_0=0$$

$$\frac{-7}{6}c_0+\frac{1}{6}c_1+\frac{5}{6}c_2=\frac{3}{2}e^{0.5}-\frac{1}{2}$$

$$\frac{-1}{3}c_0-\frac{7}{3}c_1+\frac{5}{3}c_2=2e-1$$

Solve the three equation by Gauss elimination to find the values c_0, c_1 *and* c_2.
To get

$$c_0=0$$

$$c_1=-0.1839$$

$$c_2=2.4045$$

Then the solution of linear Fredholm integro-differential equation of the second kind is:

$$y(x)=$$

$$(c_0-2c_1+c_2)x^2-2((c_0-c_1)x+c_0$$

$$y(x)=2.7723x^2-0.3678x$$

Approximated solution for some values of (x) by using Bernstein polynomials method and exact values $y(x)=xe^{x}$ of Example, depending on least square error (L.S.E),

$$Error=\sum_{k=1}^{m}(y_{Exact}(x)-y_{Approximation}(x))^2$$

are presented in Table(1) and figure(1).

6. Conclusions

Integro-differential equations are usually difficult to solve analytically. It is required to obtain the approximate solutions.

In This paper presents the use of the Bernstein polynomials method, for solving linear Fredholm integro-differential equation of the second kind. From solving numerical example the following points have been identified:

1. This method can be used to solve the all kinds of linear Fredholm integro-differential equation.
2. It is clear that using the Bernstein polynomial basis function to approximate when the n^{th} degree of Bernstein polynomial is increases the error is decreases.

We can see also from Figure(1) that the approximation is good. And when comparisons approximation solution with exact solution that the Bernstein polynomial method is very effective and convenient.

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Table (1) The results of Example using (BPM) algorithm.

T	*Exact* $y(t)$	Approximation $y(t)$ of degree(n=1)	Approximation $y(t)$ of degree(n=2)
0	0	0	0
0.1	0.1105	0.8873	-0.0091
0.2	0.2443	1.7746	0.0373
0.3	0.4050	2.6620	0.1392
0.4	0.5967	3.5493	0.2964
0.5	0.8244	4.4366	0.5092
0.6	1.0933	5.3239	0.7773
0.7	1.4096	6.2112	1.1010
0.8	1.7804	7.0986	1.4800
0.9	2.2136	7.9859	1.9145
1	2.7183	8.8732	2.4045
L.S.E $Error=\sum_{k=1}^{10}(y_{Exact}(t)-y_{Approximation}(t))^2$		170.244	0.790595

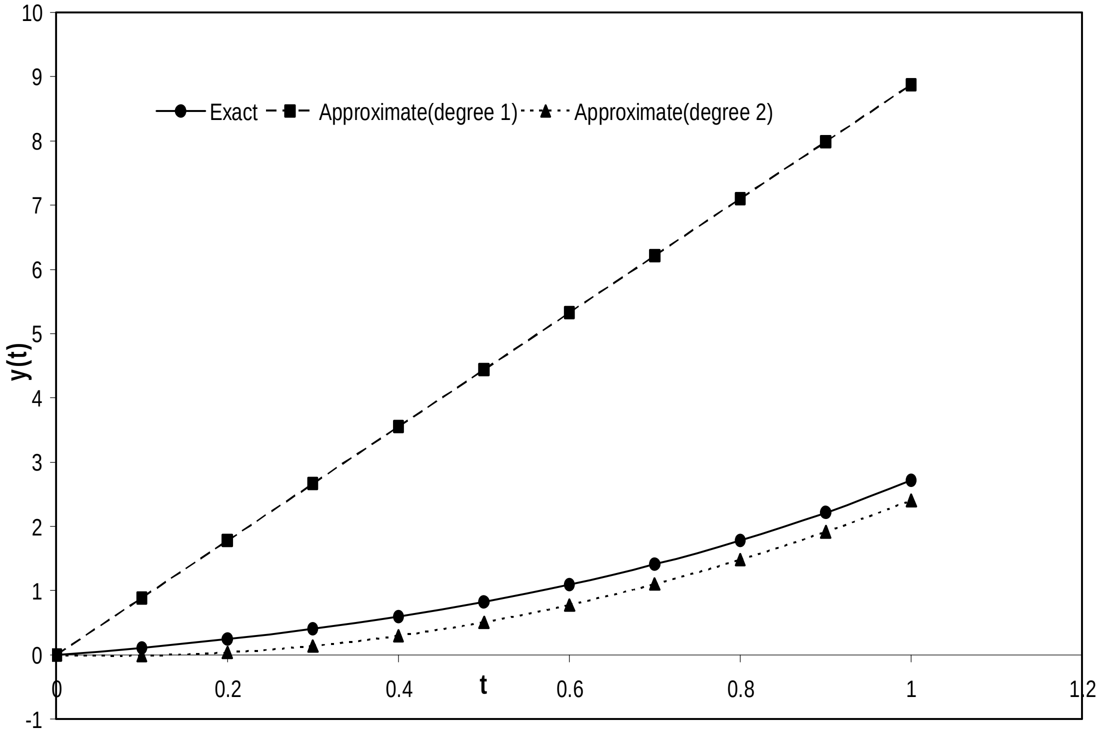

Figure (1) Approximation and Exact solution of linear Fredholm integro-differential equation of example