On Fuzzy Normed Spaces

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Abstract
In this paper we introduce a new definition of a fuzzy normed space (to the best of our knowledge) then the related concepts such as fuzzy continuous, convergence of sequence of fuzzy points and Cauchy sequence of fuzzy points are discussed in details.

Keywords: fuzzy set, fuzzy normed space, sequence of fuzzy points, fuzzy continuous.

S1: Basic Concepts About Fuzzy Sets
Definition 1.1: [1]
Let X be a nonempty set of elements. A fuzzy set $\tilde{A}$ in X is characterized by a membership function, $\mu_{\tilde{A}}: X \rightarrow [0,1].$ Then $\tilde{A}$ can be written by $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X, 0 \leq \mu_{\tilde{A}}(x) \leq 1\}$

Definition 1.2: [1]
Let $\tilde{A}$ and $\tilde{O}$ be two fuzzy sets in X then
1. $\tilde{A} \subseteq \tilde{O} \iff \mu_{\tilde{A}}(x) \leq \mu_{\tilde{O}}(x)$ for all $x \in X.$
2. $\tilde{A} = \tilde{O} \iff \mu_{\tilde{A}}(x) = \mu_{\tilde{O}}(x)$ for all $x \in X.$
3. Then complement of $\tilde{A}$ (denoted by $\tilde{A}^{c}$) is also a fuzzy set with membership function $\mu_{\tilde{A}^{c}}(x) = 1 - \mu_{\tilde{A}}(X)$ for all $x \in X.$

Definition 1.3: [1]
A fuzzy point $P_{\alpha}$ in X is a fuzzy set with membership function $\mu_{P_{\alpha}}(y) = \{ \alpha \text{ if } y = x \}
\begin{align*}
0 & \text{ otherwise} \\
\end{align*}$
For all $y \in X$ where $0 < \alpha < 1.$ We denote this fuzzy point by $x_{\alpha}$ or $(x, \alpha).$

Definition 1.4: [2]
Two fuzzy points $x_{\alpha}$ and $y_{\beta}$ are said to be equal if $x = y$ and $\alpha = \beta$ where $\alpha, \beta \in (0,1].$

Definition 1.5: [1]
Let $x_{\alpha}$ be a fuzzy point and $\tilde{A}$ a fuzzy set in X, then $x_{\alpha}$ is said to be in $\tilde{A}$ or belongs to $\tilde{A}$ denoted by $x_{\alpha} \in \tilde{A}$ if $\alpha \leq \mu_{\tilde{A}}(x).$

Definition 1.6: [3]
Let $f$ be a function from a

Proposition 2.2:
Let $(X, ||||)$ be an ordinary normed

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1790

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Proposition 1.7: Let \( f^{-1}(\emptyset) \) be a fuzzy set in \( X \) with membership function \( \mu_{f^{-1}(\emptyset)} = \emptyset \).
If \( A \) is a fuzzy set in \( X \) then \( f(A) \) is a fuzzy set in \( Y \) with membership
\[
\mu_{f(A)}(y) = \{ \sup \{ \mu_A(x) \mid x \in f^{-1}(y) \} \text{ if } f^{-1}(y) \neq \emptyset \\
0 \text{ otherwise}
\]
For all \( y \in Y \) where \( f^{-1}(y) = \{ x \in X \mid f(x) = y \} \).

**Proposition 1.7:** [4]

Let \( f: X \to Y \) be a function then for a fuzzy point \( x_\alpha \) in \( X \), \( f(x_\alpha) \) is a fuzzy point in \( Y \) and \( f(x_\alpha) = f(x_\beta) \).

**Definition 2.1:** [2]

Let \( X \) be a vector space over field \( \mathbb{K} \) and let \( A \) be a fuzzy set in \( X \).
Then \( A \) is called a fuzzy subspace of \( X \) if for all \( x, y \in X \) and \( \lambda \in \mathbb{K} \).

(i) \( \mu_\lambda(x+y) \geq \min\{\mu_\lambda(x), \mu_\lambda(y)\} \)

(ii) \( \mu_\lambda(\lambda x) \geq \mu_\lambda(x) \)

**S2: Fuzzy Normed Spaces**

**Definition 2.1:**

Let \( X \) be a vector space over field \( \mathbb{K} \) (\( \mathbb{K} = \mathbb{R} \) or \( \mathbb{K} = \mathbb{C} \)). Let \( \| \cdot \| : X \to [0,\infty) \) be a function which assigns to each point \( x_\alpha \) in \( X \) \( \alpha \in (0,1] \) a nonnegative real number \( \| x_\alpha \| \) such that

(\( FN_1 \)) \( \| x_\alpha \| = 0 \) if and only if \( x = 0 \)

(\( FN_2 \)) \( \lambda \| x_\alpha \| = |\lambda| \| x_\alpha \| \) for all \( \lambda \in \mathbb{K} \)

(\( FN_3 \)) \( \| x_\alpha + y_\beta \| \leq \| x_\alpha \| + \| y_\beta \| \)

(\( FN_4 \)) if \( \| x_\alpha \| < r \) where \( r > 0 \) then there exists

\[ 0 < \gamma \leq \alpha < 1 \] such that \( \| x_\gamma \| < r \).

Then \( \| \cdot \| \) is called a fuzzy norm and \( (X, \| \cdot \|) \) is called a fuzzy normed space.

space, define \( \| x_\alpha \| = \frac{1}{\alpha} \| x \| \) for every \( x_\alpha \in X \) where \( \alpha \in (0,1] \). Then \( (X, \| \cdot \|) \) is a fuzzy normed space.

**Proof:** Let \( x_\alpha, y_\beta \in X \) where \( \alpha, \beta \in (0,1] \) and \( \gamma \in \mathbb{K} \).

(\( FN_1 \)) \( \| x_\alpha \| = 0 \iff \frac{1}{\alpha} \| x \| = 0 \iff \| x \| = 0 \iff x = 0 \)

(\( FN_2 \)) \( \| \gamma x_\alpha \| = \frac{1}{\alpha} \| \gamma x \| = |\gamma| \| x_\alpha \| \)

(\( FN_3 \)) \( \| x_\alpha + y_\beta \| = \| (x+y)_\gamma \| \) \( \gamma = \max\{\alpha, \beta\} \)

\[ = \frac{1}{\alpha} \| x + y \| \]

\[ \leq \frac{1}{\alpha} \| x \| + \frac{1}{\beta} \| y \| \]

\[ \leq \| x_\alpha \| + \| y_\beta \| \]

(\( FN_4 \)) if \( \| x_\alpha \| < r \) where \( r > 0 \) then

for \( \sigma \in (0,1] \) with \( \alpha \leq \sigma \), we have \( \| x_\alpha \| \leq \frac{\| x \|}{\sigma} < r \) that is \( \| x_\alpha \| < r \)

The proof of the following result is clear. Hence is omitted.

**Proposition 2.3:**

Let \( (X, \| \cdot \|) \) be a fuzzy normed space by defining

\[ \| x \| = \| (x,1) \| \], then \( (X, \| \cdot \|) \) is a normed spaces .

**Example 2.4:**

Let \( X = \mathbb{R} \), then \( \| x_\alpha \| = \frac{1}{\alpha} \| x \| \) is a fuzzy norm on \( \mathbb{R} \) by proposition 2.2 called the usual fuzzy norm.

**Remark 2.5:**

From the definition 2.1 we obtain by induction the generalized of

(\( FN_3 \)) \( \| (x_1, \alpha_1) - (x_\alpha, \alpha_\alpha) \| \leq \| (x_1, \alpha_1) - (x_\alpha, \alpha_\alpha) \| \)

\[ = \| (x_\alpha, \alpha_\alpha) \| + \| (x_1, \alpha_1) - (x_\alpha, \alpha_\alpha) \| + \ldots + \| (x_n, \alpha_n) - (x_n, \alpha_n) \| \]

Where \( (x_\alpha, \alpha_\alpha), (x_1, \alpha_1), \ldots, (x_n, \alpha_n) \in X \)
**Definition 2.6:**
A fuzzy subspace \( \hat{Y} \) of a fuzzy normed space \((X, ||||)\) is a fuzzy subspace of \(X\) considered as a vector space with the fuzzy norm obtained by restricting the fuzzy norm on \(X\) to \(\hat{Y}\).

**Definition 3.1:**
Let \((X, ||||)\) be a fuzzy normed space. Given \(x_0 \in X\), where \(\alpha \in (0,1]\) and a real number \(r > 0\)

(i) \(\hat{O}(x_0, r) = \{y \in X : ||x_0 - y|| < r\}\) is open fuzzy ball, where \(\beta \in (0,1]\)

(ii) \(\hat{O}(x_0, r) = \{y \in X : ||x_0 - y|| \leq r\}\) is closed fuzzy ball, where \(\beta \in (0,1]\)

(iii) \(S(x_0, r) = \{y \in X : ||x_0 - y|| = r\}\) is fuzzy sphere, where \(\beta \in (0,1]\)

In all three cases, \(x_0\) is called the center and \(r\) is radius.

**Definition 3.2:**
A fuzzy set \(\hat{A}\) in fuzzy normed space \((X, ||||)\) is said to be open if it contains a fuzzy ball about each of it’s fuzzy element.

A fuzzy set \(\hat{O}\) is said to be closed if it’s complement is open fuzzy set.

**Definition 3.3:**
Let \((X, ||||)\) be a fuzzy normed space, an open fuzzy ball \(\hat{O}(x_0, \epsilon)\) of radius \(\epsilon\) is often called an \(\epsilon\) - neighborhood of \(x_0\) (here \(\epsilon > 0\)).

By a neighborhood of \(x_0\) we mean a fuzzy set of \(X\) which contains an \(\epsilon\) - neighborhood of \(x_0\).

**Definition 3.4:**
We call \(x_0\) an interior fuzzy point of the the fuzzy set \(\hat{A}\) if \(\hat{A}\) is a neighborhood of \(x_0\).

The interior of \(\hat{A}\) is the set of all interior fuzzy points of \(\hat{A}\) and is denoted by \(\text{int} (\hat{A})\).
\( \tilde{N}_2 \). Consequently \( T \) is fuzzy continuous at \( x_0 \). Since \( x_n \in X \) was arbitrary \( T \) is fuzzy continuous.

**Definition 3.7:**
Let \( \tilde{A} \) be a fuzzy set in a fuzzy normed space \((X, \| \|_{\tilde{F}})\). Then a fuzzy point \( x_0 \in X \) where \( \alpha \in (0,1] \) (which may or not be a fuzzy element of \( \tilde{A} \)) is called a limit of \( \tilde{A} \) if every neighborhood of \( x_0 \) contains at least one fuzzy element \( y_0 \in \tilde{A} \) distinct from \( x_0 \). The fuzzy set consisting of \( \tilde{A} \) and its limit fuzzy points is called closure of \( \tilde{A} \) and is denoted by cl \((\tilde{A})\). It is the smallest closed fuzzy set containing \( \tilde{A} \).

**Definition 3.8:**
A fuzzy set \( \tilde{A} \) of a fuzzy normed space \((X, \| \|_{\tilde{F}})\) is said to be dense in \( X \) if cl \((\tilde{A})\) = \( X \).

**S4: Convergence, Cauchy Fuzzy Sequences**

**Definition 4.1:**
A fuzzy sequence \( \{(x_n, \alpha_n)\} \) in a fuzzy normed space \((X, \| \|_{\tilde{F}})\) is said to be convergent to \( x_0 \) in \( X \) where \( \alpha_1, \alpha_2 \in (0,1] \) for \( i=1,2, \ldots \) if
\[
\lim_{n \to \infty} \| (x_n, \alpha_n) - x_0 \|_{\tilde{F}} = 0
\]
x_0 is called the limit of \( \{(x_n, \alpha_n)\} \) and we write
\[
\lim_{n \to \infty} (x_n, \alpha_n) = x_0 \text{ or simply } (x_n, \alpha_n) \to x_0
\]
if \( \{(x_n, \alpha_n)\} \) is not convergent then it is called divergent.

**Remark 4.2:**
If \( (x_n, \alpha_n) \to x_0 \), an \( \varepsilon > 0 \) being given, there is a positive integer \( N \) such that \( (x_n, \alpha_n) \) with \( n>N \) lie in \( \varepsilon \)-neighborhood \( B(x_0, \varepsilon) \) of \( x_0 \), that is:
\[
\| (x_n, \alpha_n) - x_0 \|_{\tilde{F}} < \varepsilon \text{ for all } n\geq N
\]

**Definition 4.3:**
We call a nonempty fuzzy set \( \tilde{A} \) in \((X, \| \|_{\tilde{F}})\) bounded if its fuzzy diameter
\[
\delta(\tilde{A}) = \sup \{ \| x_0 - y \|_{\tilde{F}} : x_0, y \in \tilde{A} \}
\]
into \( \tilde{N}_2 \) because \( \tilde{N}_1 \) is mapped into

**Definition 4.4:**
In a fuzzy normed space \((X, \| \|_{\tilde{F}})\) we call a sequence \( \{(x_n, \alpha_n)\} \) is bounded if the corresponding fuzzy set is bounded.

**Remark 4.5:**
If \( \tilde{A} \) is a bounded fuzzy set then \( \tilde{A} \subset \tilde{O}(x_n, r) \) where \( x_n \in \tilde{A} \) is any fuzzy element and \( r>0 \) is a (sufficiently large) real number.

**Theorem 4.6:**
Let \((X, \| \|_{\tilde{F}})\) be a fuzzy normed space. Then (i) a convergent fuzzy sequence in \( X \) is bounded and its limit is unique.

(ii) if \( (x_n, \alpha_n) \to x_0 \) and \( (y_m, \beta_m) \to y_0 \) in \( X \)
There \( \alpha, \alpha, \beta, \beta, \in (0,1] \) i= 1, 2, \ldots .
Then \( \| (x_n, \alpha_n) - (y_m, \beta_m) \|_{\tilde{F}} \to 0 \) \( x_0 = y_0 \)

**Proof:**
(i) Suppose that \( (x_n, \alpha_n) \to x_0 \) then taking \( \varepsilon = 1 \) we can find \( N > 0 \) such that \( \| (x_n, \alpha_n) - x_0 \|_{\tilde{F}} < 1 \) for all \( n\geq N \).

Hence by remark 2.5 for all \( n \) we have
\[
\| (x_n, \alpha_n) - x_0 \|_{\tilde{F}} < 1 + a, \text{ where } a = \max \{ \| (x_i, \alpha_i) - x_0 \|_{\tilde{F}}, \| (x_2, \alpha_2) - x_0 \|_{\tilde{F}}, \ldots \} \}
\]
This shows that \( \{(x_n, \alpha_n)\} \) is bounded. Now, assuming that \( (x_n, \alpha_n) \to x_0 \) and
\( (x_n, \alpha_n) \to z_\beta \), we obtain from (FN3)
\[0 \leq \| x_0 - z_\beta \|_{\tilde{F}} \leq \| x_0 - (x_n, \alpha_n) \|_{\tilde{F}} + \| (x_n, \alpha_n) - z_\beta \|_{\tilde{F}} \]
Thus \( \| x_0 - z_\beta \|_{\tilde{F}} = 0 \) which implies that \( x_0 = z_\beta \).

(ii) By remark 2.5 we have
\[
\| (x_n, \alpha_n) - (y_m, \beta_m) \|_{\tilde{F}} \leq \| (x_n, \alpha_n) - x_0 \|_{\tilde{F}} + \| x_0 - y_m \|_{\tilde{F}} + \| y_m - (y_m, \beta_m) \|_{\tilde{F}}
\]
Hence we obtain
\[
\| (x_n, \alpha_n) - (y_m, \beta_m) \|_{\tilde{F}} - \| x_0 - y_m \|_{\tilde{F}} \leq \| (x_n, \alpha_n) - x_0 \|_{\tilde{F}} + \| (y_m, \beta_m) - y_m \|_{\tilde{F}}
\]
And a similar inequality by interchanging \( (x_n, \alpha_n) \) and \( x_n \) as well.
Theorem 4.9:
Every convergent fuzzy sequence in a fuzzy normed space \((X, ||.||)\) is Cauchy.

Proof:
Let \((x_n, \alpha_n)\) be a sequence of fuzzy points in \(X\) such that \((x_n, \alpha_n) \to x_\alpha\) for every \(\varepsilon > 0\) there is an integer \(N > 0\) such that \(|| (x_n, \alpha_n) - x_\alpha || < \varepsilon\) for all \(n > N\).

Hence by (FN3) we obtain for \(m, n > N\),
\[
|| (x_m, \alpha_m) - (x_n, \alpha_n) || \leq || (x_m, \alpha_m) - x_\alpha || + || x_\alpha - (x_n, \alpha_n) || \leq \varepsilon + \frac{\varepsilon}{2} = \varepsilon/2
\]

This shows that \(\{(x_n, \alpha_n)\}\) is Cauchy.

Theorem 4.9:
Let \(\tilde{A}\) be a nonempty fuzzy set in a fuzzy normed space \((X, ||.||)\) and \(\text{cl}(\tilde{A})\) its closure. Then

(i) \(x_\alpha \in \text{cl}(\tilde{A})\) if and only if there is a fuzzy sequence \(\{(x_n, \alpha_n)\}\) in \(\tilde{A}\) such that \((x_n, \alpha_n) \to x_\alpha\).

(ii) \(\tilde{A}\) is closed if and only if the situation \((x_n, \alpha_n) \in \tilde{A}\) and \((x_n, \alpha_n) \to x_\alpha\) implies \(x_\alpha \in \tilde{A}\).

Proof:
(i) Let \(x_\alpha \in \text{cl}(\tilde{A})\). If \(x_\alpha \notin \tilde{A}\) a fuzzy sequence of that type is \(x_n, x_\alpha,\ldots\). If \(x_n \notin \tilde{A}\) it is a limit of \(\tilde{A}\). Hence as \((y_n, \beta_n)\) and \(y_\beta\) and multiplying

\((x_n, \alpha_n) \to x_\alpha\) because \(\frac{1}{n} \to 0\) as \(n \to \infty\).

Conversely, if \(\{(x_n, \alpha_n)\}\) is in \(\tilde{A}\) and \((x_n, \alpha_n) \to x_\alpha\) then \(x_\alpha \in \tilde{A}\) or every neighborhood of \(x_\alpha\) contains a fuzzy point \((x_n, \alpha_n) \neq x_\alpha\) so that is \(x_\alpha\) a limit of \(\tilde{A}\). Hence \(x_\alpha \in \text{cl}(\tilde{A})\). It is clear that \(\tilde{A} = \text{cl}(\tilde{A})\).

(ii) \(\tilde{A}\) is closed if and only if \(\tilde{A} = \text{cl}(\tilde{A})\) so that (ii) follows readily from (i).

Theorem 4.10:
A mapping \(T:X \to Y\) of a fuzzy normed space \((X, ||.||_{f_1})\) into a fuzzy normed space \((Y, ||.||_{f_2})\) is fuzzy continuous at a fuzzy point \(x_\alpha \in X\) if and only if \((x_n, \alpha_n) \to x_\alpha\) implies \((T(x_n), \alpha_n) \to T(x_\alpha)\).

Proof:
Assume that \(T\) is fuzzy continuous at \(x_\alpha\). Then given \(\varepsilon > 0\) there is \(\delta > 0\) such that \(|| y_\beta - x_\alpha ||_{f_1} < \delta\) implies \(|| T(y_\beta) - T(x_\alpha) ||_{f_2} < \varepsilon\). Let \((x_n, \alpha_n) \to x_\alpha\). Then there is \(N > 0\) such that \(|| (x_n, \alpha_n) - x_\alpha ||_{f_1} < \varepsilon\) for all \(n > N\).

Hence for all \(n > N\),
\[
|| (T(x_n), \alpha_n) - T(x_\alpha) ||_{f_2} < \varepsilon
\]

This means that \((T(x_n), \alpha_n) \to T(x_\alpha)\).

Conversely, we assume that \((x_n, \alpha_n) \to x_\alpha\) implies \((T(x_n), \alpha_n) \to T(x_\alpha)\), and prove that \(T\) is fuzzy continuous at \(x_\alpha\). Suppose this is false. Then there is an \(\varepsilon > 0\) such that for every \(\delta > 0\) there is a \(y_\beta \neq x_\alpha\) satisfying \(|| y_\beta - x_\alpha ||_{f_1} < \delta\) but \(|| T(y_\beta) - T(x_\alpha) ||_{f_2} \geq \varepsilon\).

In particular for \(\delta = \frac{\varepsilon}{n}\) there is \((x_n, \alpha_n)\) satisfying \(|| (x_n, \alpha_n) - x_\alpha ||_{f_1} < \frac{\varepsilon}{n}\) but \(|| (T(x_n), \alpha_n) - T(x_\alpha) ||_{f_2} \geq \varepsilon\). Clearly \((x_n, \alpha_n) \to x_\alpha\) but \(T(x_n), \alpha_n) \to T(x_\alpha)\).
for each $n = 1, 2, \ldots$ the fuzzy ball $B\left(x_n, \frac{1}{n}\right)$ contains an $(x_n, \alpha_n) \in \tilde{A}$ and $(x_n, \alpha_n)$ does not converge to $T(x_n)$. This contradicts $(T(x_n), \alpha_n) \rightarrow T(x)_\alpha$ and proves the theorem.

References: