Power Series Method For Solving Nonlinear Volterra Integro-Differential Equations of The Second Kind

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Abstract

In this work, we present the power series method for solving special types of the first order nonlinear Volterra integro-differential equations of the second kind. To show the efficiency of this method, we solve some numerical examples.

Keywords: Integro-differential, power series.

طريقة متسلسلات القوى لحل معادلات فولتيرا التكاملية - التفاضلية اللاخطية ذات الرتبة الاولى ومن النوع الثاني

الخلاصة

في هذا العمل قمنا بتقديم طريقة متسلسلات القوى لحل أنواع خاصة من معادلات فولتيرا التكاملية - التفاضلية اللاخطية ذات الرتبة الأولى ومن النوع الثاني. ولإثبات كفاءة هذه الطريقة قمنا بحل بعض الأمثلة العددية.

1. Introduction

It is known that the integro-differential equations arise in a great many branches of sciences, for example, in potential theory, acoustics, elasticity, fluid mechanics, theory of population, [4], [3].

Many researchers studied the integro-differential equations, see [4] discussed the existence of the solutions for special types of integro-differential equations, [5], devoted some analytic methods for solving linear Volterra integro-differential equations, [6], [1] gave some numerical methods for solving linear and nonlinear Volterra integro-differential equations, [5] used some approximated methods for solving linear Volterra integro-differential equations.

The power series method is one of the important methods that can be used to solve the initial value problem of the linear Volterra integro-differential equations of the second kind, [2].

In [7], the power series method is used to solve the nonlinear Volterra integral equations of the second kind of the form:

$$u(x)=f(x)+l\int_{0}^{x}k(x,t)\left[u(t)\right]^{p}dt,\ p\in \mathrm{N}$$

where f and k are known functions, λ is a scalar parameter and u is the unknown function that must be determined.

Here we use the same method to solve the initial value problem that consists of the first order non-linear Volterra integro-differential equations of the second kind of the form:

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$$u'(x) = f(x) + l\int_0^x k(x,t)[u(t)]^p\,dt,$$
$$p \in \mathrm{N} \qquad .....(1.a)$$

together with the initial condition:

$$u(0) = \alpha \qquad .... (1.b)$$

where f and k are known functions, α is a known constant, λ is a scalar parameter and u is the unknown function that must be determined.

2. Power Series Method for Solving Equations (1):

Consider the initial value problem given by equations (1). Assumed the solution of equations (1) takes the form:

$$u(x) \cong e_0 + e_1 x + e_2 x^2 \ ... (2)$$

Then by setting x=0 into equation (2) one can get:

$$u(0) \cong e_0.$$

By using the initial condition given by equation (1.b), one can get:

$$e_0 = \alpha.$$

Then by differentiating equation (2) with respect to x and setting x=0 in the resulting equation on can have:

$$u'(0) = e_1.$$

On the other hand, from equation (1.a), one can have:

$$u'(0) = f(0).$$

Therefore

$$e_1 = f(0).$$

Thus the approximated solution takes the form:

$$u(x) \cong \alpha + f(0)x + e_2 x^2 \quad .... (3)$$

where e_2 is the unknown parameter that must be determined. To do this, we expand k(x,y) and f(x) as a power series. That is,

$$k(x,t) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} k_{ij}\, x^i t^j \quad ..... (4)$$

and

$$f(x) = \sum_{i=0}^{\infty} f_i x^i \qquad ..... (5)$$

By substituting equations (3)-(5 into equation (1.a) one can get:

$$f(0) + 2e_2 x = \sum_{i=0}^{\infty} f_i x^i + \int_0^x \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} k_{ij} x^i t^j \left[a + f(0)t + e_2 t^2\right]^p dt$$
$$.... (6)$$

But

$$\left[a + f(0)t + e_2 t^2\right]^p =$$

$$\sum_{k=0}^{p} \binom{p}{k} a^k \left[f(0)t + e_2 t^2\right]^{p-k} =$$

$$\sum_{k=0}^{p} \binom{p}{k} a^k t^{p-k} \left[f(0) + e_2 t\right]^{p-k} =$$

$$\sum_{k=0}^{p} \binom{p}{k} a^k t^{p-k} \sum_{\mathbf{l}=0}^{p-k} \binom{p-k}{\mathbf{l}}$$

$$[f(0)]^{\mathbf{l}} [e_2 t]^{p-k-\mathbf{l}}$$

Therefore equation (6) becomes

$$f(0) + 2e_2 x = \sum_{i=0}^{\infty} f_i x^i +$$

$$\int_0^x \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} k_{ij} x^i t^j \sum_{k=0}^{p} \binom{p}{k} a^k t^{p-k}$$

$$\sum_{\mathbf{l}=0}^{p-k} \binom{p-k}{\mathbf{l}} [f(0)]^{\mathbf{l}} [e_2 t]^{p-k-\mathbf{l}} dt =$$

$$f_0 + f_1 x + f_2 x^2 + \mathbf{L} +$$

$$\int_0^x (k_{00} + k_{10} x + k_{01} t + \mathbf{L})$$

$$\left[\binom{p}{0} t^p \sum_{\mathbf{l}=0}^{p} \binom{p}{\mathbf{l}} [f(0)]^{\mathbf{l}} [e_2 t]^{p-\mathbf{l}} +\right.$$

$$\binom{p}{1} a\, t^{p-1} \sum_{\mathbf{l}=0}^{p-1} \binom{p-1}{\mathbf{l}} [f(0)]^{\mathbf{l}}$$

$$[e_2 t]^{p-1-\mathbf{l}} + \binom{p}{2} a^2 t^{p-1}$$

$$\sum_{\mathbf{l}=0}^{p-1} \binom{p-2}{\mathbf{l}} [f(0)]^{\mathbf{l}} [e_2 t]^{p-2-\mathbf{l}} + \mathbf{L}$$

$$\binom{p}{p} a^p\, t^{p-p} \sum_{\mathbf{l}=0}^{p-p} \binom{p-p}{\mathbf{l}}$$

$$\left.[f(0)]^{\mathbf{l}} [e_2 t]^{p-p-\mathbf{l}}\right] dt$$

$$f_0+f_1x+f_2x^2+\mathbf{L}+ \int_0^x (k_{00}+k_{10}x+k_{01}t+\mathbf{L})\Big[t^p\Big\{(e_2t)^p+ \binom{p}{1}f(0)(e_2t)^{p-1}+ \binom{p}{2}[f(0)]^2(e_2t)^{p-2}+\mathbf{L}+ [f(0)]^p\Big\}+pa\,t^{p-1}\Big\{(e_2t)^{p-1}+ \binom{p-1}{1}f(0)(e_2t)^{p-1}+ \binom{p-1}{2}[f(0)]^2(e_2t)^{p-3}+\mathbf{L}+ [f(0)]^{p-1}\Big\}+\frac{p(p-1)}{2}a^2t^{p-2} \Big\{(e_2t)^{p-2}+\binom{p-2}{1}f(0)(e_2t)^{p-3}+ \binom{p-2}{2}[f(0)]^2(e_2t)^{p-4}+\mathbf{L}+ [f(0)]^{p-2}\Big\}+a^p\Big]dt$$

$$Q(x^2)=f_2x^2+f_3x^3+\mathbf{L}+ \int_0^x (k_{00}+k_{10}x+k_{01}t+\mathbf{L})\Big[t^p\Big\{(e_2t)^p+ \binom{p}{1}f(0)(e_2t)^{p-1}+ \binom{p}{2}[f(0)]^2(e_2t)^{p-2}+\mathbf{L}+ [f(0)]^p\Big\}+pa\,t^{p-1}\Big\{(e_2t)^{p-1}+ \binom{p-1}{1}f(0)(e_2t)^{p-1}+ \binom{p-1}{2}[f(0)]^2(e_2t)^{p-3}+\mathbf{L}+ [f(0)]^{p-1}\Big\}+\frac{p(p-1)}{2}a^2t^{p-2} \Big\{(e_2t)^{p-2}+\binom{p-2}{1}f(0)(e_2t)^{p-3}+ \binom{p-2}{2}[f(0)]^2(e_2t)^{p-4}+\mathbf{L}+ [f(0)]^{p-2}\Big\}$$

It is clear that $f(0)=f_o$ and $Q(x^2)$ is a polynomial of degree greater than or equal two. By neglecting $Q(x^2)$ and solving the equation $e_2=\frac{f_1+k_{00}\alpha^p}{2}$, the unknown parameter e_2 is determined and therefore the coefficient of x^2 in equation (3) is obtained.

By repeating the above procedure m-1 iterations, a power series of the following form derives:

$$y(x)=\sum_{i=0}^{m} e_i x^i \qquad \text{…… (7)}$$

Equation (7) is an approximated solution of the initial value problem given by equations (1).

3. Numerical Examples:

In this section we present two examples that are solved by using power series method. These examples shows the efficiency of this method.

Example (1):

Consider the first order nonlinear integro-differential equation of the second kind:

$$u'(x) = -e^{-x} + \frac{1}{3}\left(x^2 + x + \frac{1}{3}\right)e^{-3x} - \frac{1}{3}x^2 - \frac{1}{9} + \int_0^x (x^2 + t)[u(t)]^3 dt \quad \text{....(8.a)}$$

together with the initial condition:

u(0)=1 .....(8.b)

Here

$$f(x) = -e^{-x} + \frac{1}{3}\left(x^2 + x + \frac{1}{3}\right)e^{-3x} - \frac{1}{3}x^2 - \frac{1}{9}, \ p = 3$$

and

$$k(x,t) = x^2 + t.$$

We solve this example by using the power series method. To do this, let $e_0 = u(0)$ and $e_1 = u'(0)$. Therefore $e_0 = 1$ and $e_1 = f(0) = -1$. Assume the solution of the above initial value problem takes the form:

$$u(x) \cong e_0 + e_1 x + e_2 x^2.$$

Hence

$$u(x) \cong 1 - x + e_2 x^2.$$

But

$$e^{-x} = \sum_{i=0}^{\infty} \frac{(-x)^i}{i!}$$

and

$$e^{-3x} = \sum_{i=0}^{\infty} \frac{(-3x)^i}{i!}.$$

Therefore

$$f(x) = -\sum_{i=0}^{\infty} \frac{(-x)^i}{i!} + \frac{1}{3}\left(x^2 + x + \frac{1}{3}\right) \sum_{i=0}^{\infty} \frac{(-3x)^i}{i!} - \frac{1}{3}x^2 - \frac{1}{9}$$

$$= -\left(1 - x + \frac{x^2}{2!} - \mathbf{L}\right) + \frac{1}{3}\left(x^2 + x + \frac{1}{3}\right)\left(1 - 3x + \frac{9x^2}{2!} - \mathbf{L}\right) - \frac{1}{3}x^2 - \frac{1}{9}$$

$$= \left(-1 + \frac{1}{9} - \frac{1}{9}\right) + \left(1 + \frac{1}{3} - \frac{1}{3}\right)x - \sum_{i=2}^{\infty} \frac{(-x)^i}{i!} + \frac{1}{3}\left(x^2 + x + \frac{1}{3}\right) \sum_{i=2}^{\infty} \frac{(-3x)^i}{i!} - x^3 - x^2$$

$$= -1 + x - \sum_{i=2}^{\infty} \frac{(-x)^i}{i!} + \frac{1}{3}\left(x^2 + x + \frac{1}{3}\right) \sum_{i=2}^{\infty} \frac{(-3x)^i}{i!} - x^3 - x^2$$

Hence $f_1 = 1$. On the other hand

$$k_{ij} = \begin{cases} 1 & for\,(i,j)=(0,1)\,and \\ & (i,j)=(0,2) \\ 0 & e.w \end{cases}$$

Thus $k_{00}=0$. Therefore

$$e_2 = \frac{f_1 + k_{00}\alpha^p}{2} = \frac{1}{2}.$$

In this case

$$Q(x^2) = \frac{1}{8}(e_2)^3x^8 - \frac{1}{7}(e_2)^3x^9 - \frac{6}{35}(e_2)^2x^7 + \frac{1}{2}(e_2)^2x^8 - \frac{1}{2}(e_2)^2x^6 + e_2x^6 + \frac{1}{5}e_2x^5 - \frac{4}{5}x^5 - \frac{3}{5}e_2x^7 - \frac{3}{4}e_2x^4 + \frac{3}{4}x^4 + \frac{1}{4}x^6 - \frac{1}{6}x^2 + 1 + \frac{x^2}{2!} - \frac{x^3}{3!} + \mathbf{L} - \frac{1}{3}x^2\left[1 - 3x + \frac{9}{2!}x^2 - \frac{27}{3!}x^3 + \mathbf{L}\right] - \frac{1}{3}x\left[-3x + \frac{9}{2!}x^2 - \frac{27}{3!}x^3 + \mathbf{L}\right] - \frac{1}{9}\left[\frac{9}{2!}x^2 - \frac{27}{3!}x^3 + \mathbf{L}\right] + \frac{1}{3}x^2.$$

Thus

$$u(x) \cong 1 - x + \frac{1}{2}x^2.$$

By repeating the above argument for the approximated solution:

$$u(x) \cong 1 - x + \frac{1}{2}x^2 + e_3x^3$$

one can get:

$$\left(3e_3 - \frac{1}{6} - \frac{1}{3} + 1\right)x^2 + Q(x^3) = 0 \qquad \text{….(9)}$$

where

$$Q(x^3) = \frac{12}{35}x^7 + \frac{5}{8}x^6 + \frac{7}{64}x^8 - \frac{1}{56}x^9 + \frac{3}{8}x^4 - \frac{7}{10}x^5 - \frac{1}{11}(e_3)^3x^{11} - \frac{2}{21}(e_3)^2x^9 + \frac{29}{84}e_3x^9 - \frac{1}{6}(e_3)^2x^{11} - \frac{1}{10}(e_3)^3x^{12} + \frac{9}{40}(e_3)^2x^{10} - \frac{5}{8}e_3x^8 - \frac{3}{8}(e_3)^2x^8 - \frac{3}{32}e_3x^{10} + \frac{12}{35}e_3x^7 - \frac{3}{5}e_3x^5 + \frac{1}{4}e_3x^6 + \left[-\frac{x^2}{3!} + \frac{x^4}{4!} - \mathbf{L}\right] - \frac{1}{3}x^2\left[-3x + \frac{9}{2!}x^2 - \frac{27}{3!}x^3 + \mathbf{L}\right] - \frac{1}{3}x\left[\frac{9}{2!}x^2 - \frac{27}{3!}x^3 + \mathbf{L}\right] - \frac{1}{9}\left[-\frac{27}{3!}x^3 + \frac{81}{4!}x^4 - \mathbf{L}\right].$$

By neglecting $Q(x^3)$ then equation (9) becomes

$$\left(3e_3 - \frac{1}{3} + 1 - \frac{1}{2} + \frac{1}{3}\right)x^2 = 0$$

and hence $e_3 = -\frac{1}{3!}$. Thus

$$u(x) \cong 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3.$$

By repeating the above argument for the approximated solution:

$$u(x) \cong 1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + e_4x^4$$

one can get:

$$\left(4e_4-\frac{1}{6}+1-\frac{3}{2}+\frac{1}{2}\right)x^3+$$
$$Q(x^4)=0 \qquad \text{.... (10)}$$

Where

$$Q(x^4)=\frac{59}{765}x^9+\frac{7}{320}x^{10}+\frac{5}{1188}x^{11}+\frac{13}{64}x^8+\frac{3}{8}x^4-\frac{1}{2160}x^{12}-\frac{2}{5}x^7-\frac{1}{24}(e_4)^2x^{14}+\frac{7}{12}x^6-\frac{3}{5}x^5-\frac{1}{14}(e_4)^3x^{14}-\frac{26}{63}e_4x^9+\frac{9}{35}e_4x^7-\frac{1}{13}(e_4)^3x^{15}-\frac{14}{143}(e_4)^2x^{13}-\frac{1}{2}e_4x^6-\frac{1}{4}e_4x^8-\frac{7}{40}(e_4)^2x^{12}+\frac{31}{720}e_4x^{12}-\frac{1}{132}e_4x^{13}-\frac{2}{33}(e_4)^2x^{11}-\frac{59}{396}e_4x^{11}-\frac{3}{10}(e_4)^2x^{10}+\frac{13}{40}e_4x^{10}+\left[\frac{x^4}{4!}-\frac{x^5}{5!}\mathbf{L}\right]-\frac{1}{3}x^2\left[\frac{9}{2!}x^2-\frac{27}{3!}x^3+\mathbf{L}\right]-\frac{1}{3}x\left[\frac{27}{3!}x^3+\frac{81}{4!}x^4-\mathbf{L}\right]-\frac{1}{9}\left[\frac{81}{4!}x^4+\frac{243}{5!}x^5-\mathbf{L}\right].$$

By neglecting $Q(x^4)$ then equation (10) becomes

$$\left(4e_4-\frac{1}{6}+1-\frac{3}{2}+\frac{1}{2}\right)x^3=0$$

and hence $e_4=\frac{1}{4!}$. Thus

$$u(x)\cong 1-x+\frac{1}{2!}x^2-\frac{1}{3!}x^3+\frac{1}{4!}x^4.$$

By continuing in this manner, one can get:

$$u(x)\cong 1-x+\frac{1}{2!}x^2-\frac{1}{3!}x^3+\frac{1}{4!}x^4-\frac{1}{5!}x^5+\ldots=e^{-x}$$

Note that this approximated solution is the exact solution of the initial value problem given by equations (8).

Example (2):

Consider the first order nonlinear integro-differential equation of the second kind:

$$u'(x)=3x^2-\frac{1}{8}x^8\sin x+\int_0^x t\sin x[u(t)]^2dt \quad \text{...(11.a)}$$

together with the initial condition:

$$u(0)=0 \qquad \text{.....(11.b)}$$

Here $f(x)=3x^2-\frac{1}{8}x^8\sin x$, p=2 and $k(x,t)=t\sin x$.

We solve this example by using the power series method. To do this, let $e_0=u(0)$ and $e_1=u'(0)$. Therefore $e_0=0$ and $e_1=f(0)=0$. Assume the solution of the above initial value problem takes the form:

$$u(x) \cong e_0 + e_1 x + e_2 x^2$$

Hence

$$u(x) \cong e_2 x^2.$$

But

$$\sin x = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!}.$$

Therefore

$$f(x) = 3x^2 - \frac{1}{8} x^8 \sin x$$

$$= 3x^2 - \frac{1}{8} x^8 \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!}.$$

and

$$k(x,t) = t \sin x = t \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!}.$$

Hence $f_1 = 0$ and $k_{00} = 0$ and this implies that

$$e_2 = \frac{f_1 + k_{00}\alpha^p}{2} = 0.$$

In this case

$$Q(x^2) = -\frac{1}{6}(e_2)^2 x^6 \left[\sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!} \right] - 3x^2 + \frac{1}{8} x^8 \left[\sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!} \right]$$

Thus

$$u(x) \cong 0.$$

By repeating the above argument for the approximated solution:

$$u(x) \cong e_3 x^3$$

one can get:

$$(3e_3 - 3)x^2 + Q(x^3) = 0 \quad \text{.... (12)}$$

where

$$Q(x^3) = -\frac{1}{8}(e_3)^2 x^8 \left[\sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!} \right] + \frac{1}{8} x^8 \left[\sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!} \right]$$

By neglecting $Q(x^3)$ then equation (12) becomes

$$(3e_3 - 3)x^2 = 0$$

and hence $e_3 = 1$. Thus

$$u(x) \cong x^3.$$

By repeating the above argument for the approximated solution:

$$u(x) \cong x^3 + e_4 x^4$$

one can have:

$$(4e_4)x^3 + Q(x^4) = 0 \quad \text{....(13)}$$

where

$$Q(x^4) = -\left[\frac{1}{10}(e_4)^2 x^{10} + \frac{2}{9} e_4 x^9 \right] \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!}$$

By neglecting $Q(x^4)$ then equation (13) becomes

$(4e_4)x^3 = 0$

and hence $e_4 = 0$. Thus

$u(x) \cong x^3$.

By continuing in this manner, one can get:

$$u(x) \cong x^3 + 0x^4 + 0x^5 + \ldots = x^3$$

Note that this approximated solution is the exact solution of the initial value problem given by equations (11).

Remark(1):

The power series method can be also used to solve the initial value problem that consists of the first order nonlinear Volterra integro-differential equation of the second kind:

$$u'(x) = f(x) + \int_a^x k(x,t)[u(t)]^p dt \quad \text{….(14.a)}$$

together with the initial condition:

$$u(a) = \alpha \quad \text{….(14.b)}$$

To do this let $z = t - a$ then equation (14.a) becomes

$$u'(x) = f(x) + \int_0^{x-a} k(x, z+a)[u(z+a)]^p dz$$

Then by setting $s = x - a$ in the above equation one can have:

$$y'(s) = f(s+a) + \int_0^s k(s+a, z+a)[y(z)]^p dz \quad \text{…..(15.a)}$$

where $y(s) = u(s+a)$. Thus

$$y(0) = u(a) = \alpha \quad \text{….(15.b)}$$

Therefore the initial value problem given by equations (14) reduces to the initial value problem given by equations (15).

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