Dynamic Analysis of Gough-Stewart Platform Manipulator

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Abstract

A novel derivation to evaluate all the controlled forces which cause by the motors and effected along the prismatic joints on the legs of the Gough-Stewart platform manipulator based on the virtual work method is proposed in this paper. In this paper the manipulator can be considered as a multibody mechanism with rigid elements. It can be assumed that the manipulator motion was known. The aim of the dynamic analysis in this paper is to evaluate all the controlled forces which necessary to implement the manipulator programming motion.

Keywords: Gough-Stewart, Robotics, Dynamic Analysis and Manipulator

الخصائص

تم في هذا البحث استخدام طريقة جديدة و yeniءاً لحساب كل القوى المشتركة عليها التي تولدها محركات الروبوت في المفاصل الطولية في ارجل الروبوت نوع ستيوارت. لقد تم اعتماد طريقة احتساب الجهد الافتراضي بعد أن تم اعتبار الروبوت الالية ميكانيكية مكونة من عدة أجسام صلبة، وإن حركة الروبوت معلومة ما بين نقاط معلومتين. لقد تم اعتبار الأرجل بدون أي انحناءات جانبية ناتجة من جراء تأثير القوى المختلفة المؤثرة عليها. إن الهدف الرئيسي من هذا البحث هو إيجاد طريقة علمية لاحصاء جميع القوى المشتركة عليها في محركات الروبوت والتي تعتبر ضرورية لانجاز الحركة المبرمجة للروبوت بين نقاط معلومتين.
1. Introduction.
In general, Gough-Stewart platform manipulator is a six degree of freedom with two main bodies [3]. The fixed body is called the base, while another body is regarded as movable and is called the moving plate. These two bodies are connected together by six extensible legs. In this paper we assumed that every leg of the legs of the manipulator is consists of two parts connected together with a prismatic kinematic joint (p). The prismatic joints are affected under the controlled forces which cause by the motors. All the legs are connected with the base by spherical kinematic joints (s) in the points \( A_i \), while they are connected with the moving plate by spherical joints with fingers in the points \( B_i \), as shown in Figure 1.

Force and moment analysis for any robotic system are useful to choose the suitable motors for implementation the programming motion [5]. Several methods are proposed to derive dynamic equations of the Gough-Stewart manipulator.

The motivation of this paper is to derive a mathematical formulation for evaluation all the controlled forces in the prismatic kinematic joints of the Gough-Stewart platform manipulator.

2. Position equation
In order to describe the motion of the moving plate of the manipulator relative to the base, we assumed that the moving plate attached to the base as shown in Figure 1, so the position vector of the point \( A_i \) in the Global coordinate system can be written as:

\[
r_{Ai} = \begin{pmatrix} x_{Ai} \\ y_{Ai} \\ z_{Ai} \end{pmatrix}^T, i = 1,2,..,n;
\]

and the position vector of the point \( B_i \) in the local coordinates system can be written as:

\[
r_{Bi} = \begin{pmatrix} x_{Bi} \\ y_{Bi} \\ z_{Bi} \end{pmatrix}^T, i = 1,2,..,n.
\]

The rotation of transformation between the local (moving plate) and the global coordinate systems can be represented by Euler’s rotation matrix with \( \psi \) as pitch angle, \( \theta \) as yaw angle and \( \phi \) as a roll angle [1]. Thus \( R_{01} \) is a matrix of coordinate transformation from the local to the global coordinates can be written as:

\[
R_{01} = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}
\]

where

\[
A_{11} = \cos(\psi)\cos(\phi) - \sin(\psi)\cos(\theta)\sin(\phi),
\]

\[
A_{12} = \sin(\psi)\cos(\phi) + \cos(\psi)\cos(\theta)\sin(\phi),
\]

\[
A_{13} = \sin(\theta)\sin(\phi),
\]

\[
A_{21} = -\cos(\psi)\sin(\phi) - \sin(\psi)\cos(\theta)\cos(\phi),
\]

\[
A_{22} = -\sin(\psi)\sin(\phi) + \cos(\psi)\cos(\theta)\cos(\phi),
\]

\[
A_{23} = \sin(\theta)\cos(\phi),
\]

\[
A_{31} = \sin(\psi)\sin(\theta),
\]

\[
A_{32} = -\cos(\psi)\sin(\theta),
\]

Generally, kinematic relations of manipulator are expressed by the loop equations. In Figure 1 one of the loops are represented, and its equation can be written as a leg vector:
\[
q_i = l_i = r_0 + R_{01}^T r_{Bi} - r_{Ai}^T;
\]

where \( r_0 \) the position vector of the moving plate centre in the Global coordinate system, 

Thus:

\[
q_i = r_0^T r_{0i} + r_{Bi}^T r_{Bi} + r_{Ai}^T r_{Ai} + 2r_0^T R_{01} r_{Bi} - 2r_{Ai}^T R_{01} r_{Bi} - 2r_0^T r_{Ai};
\]

\( i = 1, 2, ..., n \)

It is assumed that, the geometrical parameters of the moving plate centre can be represented as:

\[
\rho = (x, y, z, \psi, \theta, \phi)^T,
\]

and the legs vectors as:

\[
q = (q_1, q_2, \ldots, q_6)^T.
\]

Thus the position equation will be written as:

\[
\Psi_{1i}(q, \rho) = \begin{bmatrix}
\Psi_{11}(q, x, y, z, \psi, \theta, \phi) & \Psi_{12}(q, x, y, z, \psi, \theta, \phi) & \Psi_{16}(q, x, y, z, \psi, \theta, \phi)
\end{bmatrix}^T
\]

\[
\Psi_{11}(q, \rho) = 0
\]

3. Forces and moments effected on the moving plate of the manipulator.


It can be considered that \( \bar{G}_i \) is the gravity force vector of the moving plate in the Global coordinate system, so its projector on the Global coordinate is \( G_{i(0)} = (0, 0, -m_1g)^T \) and his projector on the local coordinate system is \( G_{i(1)} = R_{1i} G_{i(0)} \).

There is a moment that caused by the gravity force, its vector relative to the beginning of the local coordinate system can be given as:

\[
\bar{M}_{01}\{G_i\} = \bar{r}_{c_1} \times \bar{G}_i
\]

3.2. Working force.

It can be assumed that the moving plate affected under a working force with a vector \( \bar{P}_H \). This force will be caused a moment with a vector relative to the beginning of the local coordinate system \( \bar{M}_{01}\{\bar{P}_H\} \).

3.3. Controlled forces (motors forces).

Every leg of the legs of the manipulator affected under a controlled force which effected along the prismatic joint (along the leg). This force causes by the motor \( Q_i \)

4. Inertia forces and moments evaluation.

The formulation of the inertia force of the moving plate of the manipulator can be written as:

\[
\Phi_i = -m_1W_{c1}
\]
where \( W_{c1} \) is the vector projector of the acceleration of the moving plate centre on the local coordinate system [7]. Inertia moment of the moving plate relative to the beginning of the local coordinate system can be obtained as:

\[
\bar{L}_{01} = -\bar{J}_{c1} \cdot \bar{e}_1 - \bar{\omega}_1 \times \bar{J}_{c1} \cdot \bar{\omega}_1 + \bar{r}_{c1} \times \bar{\Phi}_1 .
\]

(4)

In the above equation \( \bar{J}_{c1} \) - inertia tensor of the moving plate relative to its mass centre; \( \bar{\omega}_1 \) - moving plate angular velocity; \( \bar{e}_1 \) - moving plate angular acceleration and \( \bar{r}_{c1} \) - radius vector of the moving plate centre. Thus the projector of the inertia moment on the local coordinate system can be written as:

\[
L_{01} = -J_{c1} \cdot e_1 - \omega_1 \times J_{c1} \omega_1 + r_{c1} \times \Phi_1 .
\]

(5)

where

\[
\begin{bmatrix}
0 & -z_{c1} & y_{c1} \\
z_{c1} & 0 & -x_{c1} \\
-x_{c1} & y_{c1} & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & -\omega_z & \omega_y \\
\omega_z & 0 & \omega_x \\
-\omega_y & \omega_x & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & -\omega_z & \omega_y \\
\omega_z & 0 & \omega_x \\
-\omega_y & \omega_x & 0
\end{bmatrix}
\]

Similarly the inertia forces and moments for the legs of the manipulator can be written as:

\[
\Phi_s = -m_s W_{cs} ,
\]

and,

\[
\bar{L}_{05} = -\bar{J}_{cs} \cdot \bar{e}_5 - \bar{\omega}_5 \times \bar{J}_{cs} \cdot \bar{\omega}_5 + \bar{r}_{cs} \times \bar{\Phi}_5 .
\]

(6)

But we assumed that the mass, inertia forces and moments of the legs is very little, so we will neglect them in the evaluations of the present work.

5. Virtual work evaluation.

The moving plate of the manipulator affected under the total force which consists of its weight, working force and inertia force. The total force vector is

\[
\bar{P} = \bar{G}_i + \bar{P}_h + \bar{\Phi}_1 .
\]

The total force causes a total moment relative to beginning of the local coordinate system as follow:

\[
\bar{M}_{01} = \bar{M}_{01} \{ \bar{G}_i \} + \bar{M}_{01} \{ \bar{P}_h \} + \bar{L}_{01} .
\]

(7)

Thus the summation of the virtual work of the moving plate by any little transformation and orientation of the moving plate can be obtained as:

\[
\delta A_i = \bar{P}_i \cdot \delta \bar{r}_{c1} + \bar{M}_{01} \cdot \delta \bar{Y}_1 .
\]

(8)

In the above equation \( \delta \bar{r}_{c1} \) - a moving plate centre little translation, \( \delta \bar{Y}_1 \) - a moving plate little rotations and it can be obtained as follow:

From Figure 2 it can be assumed that the moving plate angular orientation relative to Global coordinate by using Euler’s angles \( \delta \psi_1, \delta \theta_1, \delta \phi_1 \). The angular orientation can be written as:
\[ \delta \bar{Y}_1 = \delta \bar{Y}_1 + \delta \bar{\theta}_1 + \delta \bar{\varphi}_1 , \quad (9) \]

Equation (9) can be written in a following form:

\[ \delta Y_1 = \Gamma_1 \delta \Delta_1 \quad (10) \]

In the above equation, \( \delta \Delta_1 = \begin{pmatrix} \delta \psi_1 \\ \delta \theta_1 \\ \delta \varphi_1 \end{pmatrix} \).

\[ \Gamma_1 = \begin{bmatrix} 
\sin \varphi_1 \sin \theta_1 & \cos \varphi_1 & 0 \\
\sin \theta_1 \cos \varphi_1 & -\sin \varphi_1 & 0 \\
\cos \theta_1 & 0 & 1 
\end{bmatrix} \]

Thus:

\[ \delta Y_1 = \bar{I} (\delta \theta_1 \cos \varphi_1 + \delta \psi_1 \sin \varphi_1 \sin \theta_1) + \bar{j}_i (\delta \theta_1 \sin \varphi_1 + \delta \psi_1 \sin \theta_1 \cos \varphi_1) + \bar{k}_i (\delta \psi_1 \cos \theta_1 + \delta \varphi_1) \]

Now, the virtual work can be derived in a new relation as follow:

\[ \delta A_i = P_i^T \delta r_{ci} + M_{0i}^T \Gamma_1 \delta \Delta_1 \quad (11) \]

If we assume that all the forces and moments effected on the moving plate can be written as:

\[ F_i = (P_{ci} \quad P_{ci} \quad P_{ci} \quad M_{0ci} \quad M_{0ci} \quad M_{0ci})^T, \]

\[ K_1 = \begin{pmatrix} 
E_3 & 0 \\
0 & \Gamma_1 
\end{pmatrix} \]

and the geometrical parameters of the moving plate centre can be represented as:

\[ \rho = (x_{ci} \ y_{ci} \ z_{ci} \ \psi_1 \ \theta_1 \ \varphi_1)^T \]

the virtual work will be written as:

\[ \delta A_i = F_i^T K_i \delta \rho \quad (12) \]

6. The controlled forces evaluation.

In the static equilibrium condition the virtual work represented in equation (12) equal to virtual work which causes by the controlled forces which can given in the following form [9]:

\[ \delta A_i = Q^T \delta q \quad (13) \]

Thus:

\[ Q = \left( \frac{\delta \rho}{\delta q} \right)^T K_i^T F_i \quad (14) \]

From equation (1) we can derive the following relation:

\[ \frac{\partial \psi_{ii}}{\partial q} \delta q + \frac{\partial \psi_{ii}}{\partial \rho} \delta \rho = 0 \]

and

\[ \frac{\delta \rho}{\delta q} = -\left( \frac{\partial \psi_{ii}}{\partial \rho} \right)^{-1} \left( \frac{\partial \psi_{ii}}{\partial q} \right). \]

Where

\[ \frac{\partial \psi_{ii}}{\partial q} = \begin{pmatrix} 
\frac{\partial \psi_{ii}}{\partial q_1} & \cdots & \frac{\partial \psi_{ii}}{\partial q_6} \\
\cdots & \cdots & \cdots \\
\frac{\partial \psi_{i6}}{\partial q_1} & \cdots & \frac{\partial \psi_{i6}}{\partial q_6} 
\end{pmatrix}, \]

\[ = diag \{2q_i\} \]
and,

\[
\frac{\partial \Psi_{\Pi}}{\partial \rho} = \begin{pmatrix}
\frac{\partial \Psi_{\Pi1}}{\partial x_1} & \cdots & \frac{\partial \Psi_{\Pi1}}{\partial \varphi_1} \\
\vdots & \ddots & \vdots \\
\frac{\partial \Psi_{\Pi6}}{\partial x_1} & \cdots & \frac{\partial \Psi_{\Pi6}}{\partial \varphi_1}
\end{pmatrix}
\]

Thus the controlled forces (14) can be formulated as follows:

\[
Q = \left[ \frac{\partial \Psi_{\Pi}}{\partial \rho} \right]^{-1} \left[ \frac{\partial \Psi_{\Pi}}{\partial q} \right]^{T} K_{T}^{T} F \quad (15)
\]

From the above equation can evaluate all the controlled forces which effected on the prismatic joints of the manipulator.

7. Example of analysis:

In this example a manipulator with the following parameters is considered:

a. Coordinates of points \( A_i \) of the base (in the Global coordinates system) are as in the matrix:

\[
r_{Ai} = \begin{pmatrix} 2 & 1414 & 0 & -1414 & -2 & -1414 & 0 & 1414 \\
0 & 1414 & 2 & 1414 & 0 & -1414 & -2 & -1414 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

b. Coordinates of points \( B_i \) in the platform coordinates systems are as in the matrix:

\[
r_{Bi} = \begin{pmatrix} 16 & 145 & -0278 & -0917 & -1575 & -31 & 0278 & 0917 \\
0 & 0676 & 1575 & -0278 & -0917 & -1575 & -31 & 0278 & 0917 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

c. It can be assumed that the platform moves from the point 1 to the point 6 throw the track points 2,3,4,5. The coordinates of the center of the platform \( P (r_0) \) and Euler angles of its orientation \( (\psi, \theta, \varphi) \) are as shown in Table1.

\[
d. \quad q_i = r_0 + A_{0i} r_{Bi} - r_{Ai}; i = 1, 2, ..., 6
\]

The results for all points are as explained in Table 2.

e. The platform mass is \( m = 15Kg \),

\[
J = \begin{pmatrix} 0.1 & 0 & 0 \\
0 & 0.1 & 0 \\
0 & 0 & 0.1
\end{pmatrix} \text{ Kg m}^2
\]

and the forces and moments vector which applied to the platform is:

\[
P = \begin{pmatrix} 0.5N & 0.25N & -0.75N & 0.5Nm & 1Nm & 1.25Nm \end{pmatrix}^T
\]

f. The movement of the platform throw the track points (1,2,3,4,5,6) with acceleration and angular velocity of point P as are follow:

\[
\begin{align*}
\dot{a}_p &= \begin{pmatrix} 0.1m/s^2 \\
0.0m/s^2 \\
0.002m/s^2 \\
0.15deg/s^2 \\
0.1deg/s^2 \\
0.01deg/s^2
\end{pmatrix} \\
\dot{\phi}_p &= \begin{pmatrix} 0.1m/s^2 \\
0.0m/s^2 \\
0.002m/s^2 \\
0.15deg/s^2 \\
0.1deg/s^2 \\
0.01deg/s^2
\end{pmatrix}
\end{align*}
\]
And \[ \omega = \begin{bmatrix} 
0.8 \text{deg/ s} \\
0.16 \text{deg/ s} \\
0.01 \text{deg/ s} 
\end{bmatrix} \]

The results of the motorized forces are as shown in the Table 3. In this table it can be found that there is a singularity position in the point 4, it can be seen that, the forces are very high in this position.

8. Conclusions

In this paper, an approach for the dynamic analysis of the Gough-Stewart platform manipulator is proposed based on the virtual work. The mathematical simulation for the robot arms in this paper is a novel method and it has been derive by the authors. The stiffness of the arms caused by the motion of the platform was neglected. The joints of the manipulator's arms have been assumed as ideal joints (without friction forces). It has been proved that all the controlled forces can be evaluated without using the reactions in the kinematic joints.

9. References


**Table (1) Platform center coordinates and its angles of orientation**

<table>
<thead>
<tr>
<th>point coord.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>X (m)</td>
<td>0.25</td>
<td>0.5</td>
<td>0.85</td>
<td>1</td>
<td>1.2</td>
<td>1.4</td>
</tr>
<tr>
<td>Y (m)</td>
<td>-0.2</td>
<td>-0.45</td>
<td>-0.8</td>
<td>-1</td>
<td>-1.3</td>
<td>-1.5</td>
</tr>
<tr>
<td>Z (m)</td>
<td>2.2</td>
<td>2.35</td>
<td>2.7</td>
<td>3</td>
<td>3.2</td>
<td>3.5</td>
</tr>
<tr>
<td>ψ°</td>
<td>-10</td>
<td>-20</td>
<td>-30</td>
<td>-30</td>
<td>-40</td>
<td>-45</td>
</tr>
<tr>
<td>θ°</td>
<td>12</td>
<td>18</td>
<td>35</td>
<td>30</td>
<td>35</td>
<td>40</td>
</tr>
<tr>
<td>φ°</td>
<td>8</td>
<td>15</td>
<td>26</td>
<td>31.95832</td>
<td>35</td>
<td>40</td>
</tr>
</tbody>
</table>

**Table (2) Legs length**

<table>
<thead>
<tr>
<th>point q_i in (m)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>q1</td>
<td>2.2664</td>
<td>2.5530</td>
<td>3.2884</td>
<td>3.6198</td>
<td>4.1003</td>
<td>4.6117</td>
</tr>
<tr>
<td>q2</td>
<td>2.6035</td>
<td>3.0504</td>
<td>3.9605</td>
<td>4.2101</td>
<td>4.7743</td>
<td>5.2914</td>
</tr>
<tr>
<td>q3</td>
<td>2.5986</td>
<td>2.9654</td>
<td>3.7631</td>
<td>3.9677</td>
<td>4.3756</td>
<td>4.7763</td>
</tr>
<tr>
<td>q4</td>
<td>2.5846</td>
<td>2.9322</td>
<td>3.5634</td>
<td>3.7983</td>
<td>4.1697</td>
<td>4.5303</td>
</tr>
<tr>
<td>q5</td>
<td>2.2393</td>
<td>2.3956</td>
<td>2.6738</td>
<td>3.1315</td>
<td>3.3431</td>
<td>3.6687</td>
</tr>
<tr>
<td>q6</td>
<td>2.0335</td>
<td>2.0585</td>
<td>2.1544</td>
<td>2.6063</td>
<td>2.7686</td>
<td>3.0855</td>
</tr>
</tbody>
</table>
Table (3) Controlled forces

<table>
<thead>
<tr>
<th>Force In (N)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q₁</td>
<td>-1.832</td>
<td>-2.760</td>
<td>12.306</td>
<td>4.58E7</td>
<td>10.507</td>
<td>5.258</td>
</tr>
<tr>
<td>Q₂</td>
<td>0.057</td>
<td>1.082</td>
<td>-13.461</td>
<td>-4.26E7</td>
<td>-12.351</td>
<td>-7.239</td>
</tr>
<tr>
<td>Q₃</td>
<td>3.943</td>
<td>4.478</td>
<td>-4.062</td>
<td>-2.49E7</td>
<td>-0.546</td>
<td>2.234</td>
</tr>
<tr>
<td>Q₄</td>
<td>-3.822</td>
<td>-4.945</td>
<td>9.763</td>
<td>3.85E7</td>
<td>4.538</td>
<td>-0.069</td>
</tr>
<tr>
<td>Q₅</td>
<td>-0.867</td>
<td>-0.874</td>
<td>3.758</td>
<td>2.34E7</td>
<td>6.022</td>
<td>3.860</td>
</tr>
<tr>
<td>∑Q²(N²)</td>
<td>36.974</td>
<td>58.619</td>
<td>556.73</td>
<td>8.25E15</td>
<td>408.367</td>
<td>126.723</td>
</tr>
</tbody>
</table>

Figure (1) Position of the robot joints
Figure (2) The rotation of the axis