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# On Projective 3-Space

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## Abstract

The purpose of this work is to give some definitions and prove some theorems on projective 3-space S=PG(3,K) over a field K.

Also, the principle of duality in S is given which state that any theorem true in the projective 3-space concerned with the points, planes and the incidence relation, the same theorem is true by interchanging "point" and "plane" whenever they occur, where as the dual of a line is a line.


**Keyword :** point, plane, duality.

## حول الفضاء الثلاثي الاسقاطي


**الخلاصة**

الغرض من هذا البحث هو لاعطاء بعض التعاريف وبرهنة بعض المبرهنات في الفضاء الثلاثي الاسقاطي S=PG(3,K) حول الحقل K، كذلك اعطاء مبدأ الثنائية في S الذي ينص على ان اي مبرهنة متحققة في فضاء ثلاثي اسقاطي تتعلق بنقاط، مستويات وعلاقة الوقوع، فان نفس المبرهنة تتحقق بتبديل "النقطة" و "المستوي" اينما يقعا، بينما ثنائي المستقيم يكون مستقيماً.


## Introduction

A projective 3–space PG(3,K) over a field K is a 3–dimensional projective space which consists of points, lines and planes with the incidence relation between them, [1].

The projective 3–space satisfies the following axioms:

**A.** Any two distinct points are contained in a unique line.

**B.** Any three distinct non-collinear points, also any line and a point not on the line are contained in a unique plane.

**C.** Any two distinct coplanar lines intersect in a unique point.

**D.** Any line not on a given plane intersects the plane in a unique point.

**E.** Any two distinct planes intersect in a unique line.

Principle of duality, [2] any properly worded valid statement in a projective 3-space concerning incidence of points and planes gives rise to a second statement obtained from the first by interchanging the words "point" and "plane". Thus the dual elements are the point and the plane with the word "line" left unchanged.

Any point in PG(3,K) has the form of a quadrable $(x_1,x_2,x_3,x_4)$, where $x_1$, $x_2$, $x_3$, $x_4$ are elements in K with the exception of the quadrable consisting of four zero elements.

Two quadrables $(x_1,x_2,x_3,x_4)$ and $(y_1,y_2,y_3,y_4)$ represent the same point if there exists $\lambda$ in $K\backslash\{0\}$ such that $(x_1,x_2,x_3,x_4) = \lambda\ (y_1,y_2,y_3,y_4)$.

Similarly, any plane in PG(3,K) has the form of a quadrable $[x_1,x_2,x_3,x_4]$, where $x_1$, $x_2$, $x_3$, $x_4$ are elements in K with the exception of the quadrable consisting of four zero elements.

Two quadrables $[x_1,x_2,x_3,x_4]$ and $[y_1,y_2,y_3,y_4]$ represent the same plane if there exists $\lambda$ in $K\backslash\{0\}$ such that $[x_1,x_2,x_3,x_4] = \lambda\ [y_1,y_2,y_3,y_4]$.

Also a point $P(x_1,x_2,x_3,x_4)$ is incident with the plane $\pi\ [a_1,a_2,a_3,a_4]$ iff

$a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 = 0$.[3]

Now, some theorems on projective 3-space PG(3,k) can be proved.

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**Theorem 1:**

Four distinct points A($x_1$, $x_2$, $x_3$, $x_4$), B($y_1$, $y_2$, $y_3$, $y_4$), C($z_1$, $z_2$, $z_3$, $z_4$), and D($w_1$, $w_2$, $w_3$, $w_4$) are coplanar iff

$$\Delta = \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix} = 0.$$

**Proof**

Let π [$u_1$,$u_2$,$u_3$,$u_4$] be a plane containing the points A, B, C, D, then

$x_1 u_1 + x_2 u_2 + x_3 u_3 + x_4 u_4 = 0$

$y_1 u_1 + y_2 u_2 + y_3 u_3 + y_4 u_4 = 0$

$z_1 u_1 + z_2 u_2 + z_3 u_3 + z_4 u_4 = 0$

$w_1 u_1 + w_2 u_2 + w_3 u_3 + w_4 u_4 = 0$

It is known from the linear algebra that this system of equations have non zero solutions for $u_1$, $u_2$, $u_3$, $u_4$ iff $\Delta = 0$. Thus the necessary and sufficient conditions for four points to be coplanar that $\Delta = 0$.

**Corollary**

If four distinct points A($x_1$, $x_2$, $x_3$, $x_4$), B($y_1$, $y_2$, $y_3$, $y_4$), C($z_1$, $z_2$, $z_3$, $z_4$), and D($w_1$, $w_2$, $w_3$, $w_4$) are collinear, then $\Delta$=0.

This follows from theorem (1) and the incidence of these points on a line of some plane.

From the principle of duality, one can prove:

**Theorem 2**

Four distinct planes A[$x_1$, $x_2$, $x_3$, $x_4$], B[$y_1$, $y_2$, $y_3$, $y_4$], C[$z_1$, $z_2$, $z_3$, $z_4$], and D[$w_1$, $w_2$, $w_3$, $w_4$] are concurrent (intersecting in one point) iff

$$\Delta = \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix} = 0$$

**Theorem 3**

The equation of the plane determined by three distinct points A($y_1$, $y_2$, $y_3$, $y_4$), B($z_1$, $z_2$, $z_3$, $z_4$), and C($w_1$, $w_2$, $w_3$, $w_4$) is

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix} =$$

$$\begin{vmatrix} y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \\ w_2 & w_3 & w_4 \end{vmatrix} x_1 + \begin{vmatrix} y_3 & y_1 & y_4 \\ z_3 & z_1 & z_4 \\ w_3 & w_1 & w_4 \end{vmatrix} x_2 +$$

$$\begin{vmatrix} y_1 & y_2 & y_4 \\ z_1 & z_2 & z_4 \\ w_1 & w_2 & w_4 \end{vmatrix} x_3 + \begin{vmatrix} y_3 & y_2 & y_1 \\ z_3 & z_2 & z_1 \\ w_3 & w_2 & w_1 \end{vmatrix} x_4 = 0$$

where ($x_1$, $x_2$, $x_3$, $x_4$) be any variable point on the plane, and it's coordinates are:

$$\left[ \begin{vmatrix} y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \\ w_2 & w_3 & w_4 \end{vmatrix}, \begin{vmatrix} y_3 & y_1 & y_4 \\ z_3 & z_1 & z_4 \\ w_3 & w_1 & w_4 \end{vmatrix}, \begin{vmatrix} y_1 & y_2 & y_4 \\ z_1 & z_2 & z_4 \\ w_1 & w_2 & w_4 \end{vmatrix}, \begin{vmatrix} y_3 & y_2 & y_1 \\ z_3 & z_2 & z_1 \\ w_3 & w_2 & w_1 \end{vmatrix} \right]$$

**Theorem 4**

The equation of the point determined by three distinct planes (non-collinear) $a$[$y_1$, $y_2$, $y_3$, $y_4$], $b$[$z_1$, $z_2$, $z_3$, $z_4$], and $c$[$w_1$, $w_2$, $w_3$, $w_4$] is

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix} =$$

$$\begin{vmatrix} y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \\ w_2 & w_3 & w_4 \end{vmatrix} x_1 + \begin{vmatrix} y_3 & y_1 & y_4 \\ z_3 & z_1 & z_4 \\ w_3 & w_1 & w_4 \end{vmatrix} x_2 + \begin{vmatrix} y_1 & y_2 & y_4 \\ z_1 & z_2 & z_4 \\ w_1 & w_2 & w_4 \end{vmatrix} x_3 + \begin{vmatrix} y_3 & y_2 & y_1 \\ z_3 & z_2 & z_1 \\ w_3 & w_2 & w_1 \end{vmatrix} x_4 = 0$$

where $[x_1, x_2, x_3, x_4]$ be any variable plane passing through the point, and it's coordinates are:

$$\left( \begin{vmatrix} y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \\ w_2 & w_3 & w_4 \end{vmatrix}, \begin{vmatrix} y_3 & y_1 & y_4 \\ z_3 & z_1 & z_4 \\ w_3 & w_1 & w_4 \end{vmatrix}, \begin{vmatrix} y_1 & y_2 & y_4 \\ z_1 & z_2 & z_4 \\ w_1 & w_2 & w_4 \end{vmatrix}, \begin{vmatrix} y_3 & y_2 & y_1 \\ z_3 & z_2 & z_1 \\ w_3 & w_2 & w_1 \end{vmatrix} \right)$$

**Notation**

If $v$ is the vector with components $(a_1, a_2, a_3, a_4)$, then the symbol $P(v)$ means that the coordinates of the point P are $(a_1, a_2, a_3, a_4)$ in a projective 3–space $S = PG(3,K)$.

**Definition 1:[3]**

The points $P_i(v_i)$, with $i = 1, \ldots, m$ are linearly dependent or independent according as the vectors $v_i$ are linearly dependent or independent.

**Definition 2:[3]**

If the points $P_1, P_2, \ldots, P_m$ are linearly dependent, then at least one of the $c_i$'s of the equation $\sum_{i=1}^{m} c_i P_i(v_i) = 0$ is not equal to zero, say $c_1$, then $P_1 = \frac{-1}{c_1}(c_2 P_2 + c_3 P_3 + \cdots + c_m P_m)$. The point $P_1$ is then said to be a linear combination of the points $P_2, P_3, \ldots, P_m$.

This definition may be dualized by replacing the word "point" by the word "plane", and the geometric meaning of linear dependence of points or planes may now be given.

**Theorem 5**

Two points (planes) are linearly dependent iff they coincide.

**Proof**

Let P and Q be any two points. If P and Q are linearly dependent, then there exist $c_1$ and $c_2$ such that $(c_1, c_2) \neq (0,0)$, $c_1 P + c_2 Q = \theta$.

If $c_1 = 0$, then $c_2 Q = \theta$.

This implies $c_2 = 0$, since $Q \neq (0,0,0)$. Then $c_1 \neq 0$ and similarly $c_2 \neq 0$, $P = \frac{-c_2}{c_1} Q$.

This means that P and Q coincide. If P and Q are coincide, then there exist $c_1 \neq 0$, $c_2 \neq 0$ s.t. $c_1 P = c_2 Q$.

Hence, $c_1 P - c_2 Q = \theta$ and thus P and Q are linearly dependent.

**Theorem 6**

Four points are linearly dependent iff they are coplanar.

**Proof**

Let $A(x_1, x_2, x_3, x_4)$, $B(y_1, y_2, y_3, y_4)$, $C(z_1, z_2, z_3, z_4)$, and $D(w_1, w_2, w_3, w_4)$ be any four points in S. If A, B, C, D are linearly dependent, then there exist $c_1$, $c_2$, $c_3$ and $c_4$ in K such that $(c_1, c_2, c_3, c_4) \neq (0,0,0,0)$ and $c_1 A + c_2 B + c_3 C + c_4 D = \theta$

$c_1 A + c_2 B + c_3 C + c_4 D = c_1 (x_1, x_2, x_3, x_4) + c_2 (y_1, y_2, y_3, y_4) + c_3 (z_1, z_2, z_3, z_4) + c_4 (w_1, w_2, w_3, w_4) = (0,0,0,0)$

$$\begin{aligned} c_1 x_1 + c_2 y_1 + c_3 z_1 + c_4 w_1 &= 0 \\ c_1 x_2 + c_2 y_2 + c_3 z_2 + c_4 w_2 &= 0 \\ c_1 x_3 + c_2 y_3 + c_3 z_3 + c_4 w_3 &= 0 \\ c_1 x_4 + c_2 y_4 + c_3 z_4 + c_4 w_4 &= 0 \end{aligned} \quad \ldots(1)$$

This system has non zero solutions for $c_1, c_2, c_3, c_4$ iff

$$\Delta = \begin{vmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \\ x_4 & y_4 & z_4 & w_4 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix} = 0$$

by theorem (1) the points A, B, C, D are coplanar.

Conversely, if the points A, B, C, D are coplanar, then

$$\Delta = \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix} = 0, \text{ then}$$

$$\begin{vmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \\ x_4 & y_4 & z_4 & w_4 \end{vmatrix} = 0, \text{ so the system}$$

(1) of equations has non zero solutions for $c_1$, $c_2$, $c_3$, $c_4$. Thus A, B, C, D are linearly dependent.

**Theorem 7**

Any five points (planes) in S are linearly dependent.

**Proof**

Let A($a_1$, $a_2$, $a_3$, $a_4$), B($b_1$, $b_2$, $b_3$, $b_4$), C($c_1$, $c_2$, $c_3$, $c_4$), D($d_1$, $d_2$, $d_3$, $d_4$) and

E($e_1$, $e_2$, $e_3$, $e_4$) be any five points in S. Let $a\,\mathrm{A} + b\,\mathrm{B} + c\,\mathrm{C} + d\,\mathrm{D} + e\,\mathrm{E} = \theta$

$a\ (a_1,a_2,a_3,a_4) + b\ (b_1,b_2,b_3,b_4) + c\ (c_1,c_2,c_3,c_4) + d\ (d_1,d_2,d_3,d_4) + e(e_1,e_2,e_3,e_4) = \theta$

$a\, a_1 + b\, b_1 + c\, c_1 + d\, d_1 + e\, e_1 = 0$

$a\, a_2 + b\, b_2 + c\, c_2 + d\, d_2 + e\, e_2 = 0$

$a\, a_3 + b\, b_3 + c\, c_3 + d\, d_3 + e\, e_3 = 0$

$a\, a_4 + b\, b_4 + c\, c_4 + d\, d_4 + e\, e_4 = 0$

This system of 4 linear homogeneous equations in 5 unknowns $a$, $b$, $c$, $d$, $e$ has non trivial solutions since 4 < 5. Then A, B, C, D, E are linearly dependent.

**Theorem 8**

If $\mathrm{P}_1$, $\mathrm{P}_2$, …, $\mathrm{P}_m$ are linearly independent points while $\mathrm{P}_1$, $\mathrm{P}_2$, …, $\mathrm{P}_{m+1}$ are linearly dependent, then the coordinates of the points may be chosen so that $\mathrm{P}_1 + \mathrm{P}_2 + \cdots + \mathrm{P}_m = \mathrm{P}_{m+1}$.

**Proof**

Since the points $\mathrm{P}_1$, $\mathrm{P}_2$, …, $\mathrm{P}_{m+1}$ are linearly dependent, constants $c_1$, $c_2$, …,$c_{m+1} \neq 0$, 0, …, 0 exist such that

$c_1\,\mathrm{P}_1(v_1) + c_2\,\mathrm{P}_2(v_2) + \cdots + c_m\,\mathrm{P}_m(v_m) + c_{m+1}\,\mathrm{P}_{m+1}(v_{m+1}) = \theta$.

Now, $c_{m+1} \neq 0$, for otherwise the points $\mathrm{P}_1$, $\mathrm{P}_2$, …, $\mathrm{P}_m$ would be dependent contrary to hypothesis. The equation may, therefore, be solved for $\mathrm{P}_{m+1}$ giving

$$\mathrm{P}_{m+1} = -\frac{1}{c_{m+1}}[\,c_1\,\mathrm{P}_1(v_1) + \cdots + c_m\,\mathrm{P}_m(v_m)\,]$$

$$= k_1\,\mathrm{P}_1(v_1) + \cdots + k_m\,\mathrm{P}_m(v_m)$$

$$= \mathrm{P}_1(k_1\, v_1) + \cdots + \mathrm{P}_m(k_m\, v_m)$$

where $k_i = \dfrac{-c_i}{c_{m+1}}$, $i$ = 1, …, $m$ or dropping the symbols $k_i\, v_i$ , $\mathrm{P}_{m+1} = \mathrm{P}_1 + \mathrm{P}_2 + \cdots + \mathrm{P}_m$.

**Theorem 9**

A point D is on the plane determined by three distinct points A, B, C iff D is a linear combination of A, B, C.

**Proof**

If D is on the plane determined by three distinct points, then A, B, C, D are coplanar. By theorem (5), they are linearly dependent, there exist constants $a$, $b$, $c$, $d$ such that not all of them are zero and $a\,\mathrm{A} + b\,\mathrm{B} + c\,\mathrm{C} + d\,\mathrm{D} = \theta$.

If $d = 0$, then $a\,\mathrm{A} + b\,\mathrm{B} + c\,\mathrm{C} = \theta$, which implies that $a = b = c = 0$, since A, B, C are linearly independent, which is a contradiction. Since any three noncollinear points in the plane are linearly independent, [3]. So $d \neq 0$, and then

$$D = (\frac{-a}{d})\ A + (\frac{-b}{d})\ B + (\frac{-c}{d})\ C$$

Thus D is a linear combination of A, B, C. Suppose D is a linear combination of A, B, C, then there exist constants $c_1$, $c_2$, $c_3$ not all of them are zero such that:

D = $c_1$ A + $c_2$ B + $c_3$ C, which implies $c_1$ A + $c_2$ B + $c_3$ C + (–1) D = θ, then it follows that A, B, C, D are linearly dependent. By theorem (5), the points A, B, C, D are coplanar.

**Theorem 10**

The points of PG(3,K) have unique forms which are (1,0,0,0), ($x$,1,0,0), ($x$, $y$,1,0), ($x$, $y$, $z$,1) for all $x$, $y$, $z$ in K.

**Proof**

Let P($x_1$, $x_2$, $x_3$, $x_4$); $x_1$; $x_2$, $x_3$, $x_4 \in$ K be any point in PG(3,K), then either $x_4 \neq 0$ or $x_4 = 0$.

If $x_4 \neq 0$, then P($x_1$, $x_2$, $x_3$, $x_4$) ≡ $P(\frac{x_1}{x_4}, \frac{x_2}{x_4}, \frac{x_3}{x_4}, 1)$ = P($x$, $y$, $z$, 1), where

$$x = \frac{x_1}{x_4},\ y = \frac{x_2}{x_4},\ z = \frac{x_3}{x_4}.$$

If $x_4 = 0$, then either $x_3 \neq 0$ or $x_3 = 0$.

If $x_3 \neq 0$, then P($x_1$, $x_2$, $x_3$, 0) ≡ $P(\frac{x_1}{x_3}, \frac{x_2}{x_3}, 1, 0)$ = P($x$, $y$, 1, 0), where

$$x = \frac{x_1}{x_3},\ y = \frac{x_2}{x_3}.$$

If $x_3 = 0$, then either $x_2 \neq 0$ or $x_2 = 0$.

If $x_2 \neq 0$, then P($x_1$, $x_2$, 0, 0) ≡ $P(\frac{x_1}{x_2}, 1, 0, 0)$ = P($x$, 1, 0, 0), where

$$x = \frac{x_1}{x_2}.$$

If $x_2 = 0$, then $x_1 \neq 0$ and P($x_1$, 0, 0, 0) ≡ $P(\frac{x_1}{x_1}, 0, 0, 0)$ = P(1, 0, 0, 0).

Similarly, one can prove the dual of theorem (10).

**Theorem 11**

The planes of PG(3,K) have unique forms which are [1,0,0,0], [$x$,1,0,0], [$x$, $y$,1,0], [$x$, $y$, $z$,1] for all $x$, $y$, $z$ in K.

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