T-Semi Connected Spaces

Bushra Kadum Awaad* & Dr. Hadi J. Mustafa**

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Abstract

In this paper, we introduce a new concept, namely T-semi connected space, where T is an operator associated with the topological T defined on a non-empty set X. Several properties of this concept are proved.

Keywords: Semi connected, T-semi connected.

1- Introduction:

In [4], the concept of semi connected set is given. In this paper, we introduce the concept of T-semi connected set and show that it generalizes the concept of semi connected set when the operator T is the identity operator.

Throughout this paper, we use the following notations: cl(A) denotes the usual closure and int(A) denotes the interior of a set A.

2- Basic Definitions and Results:

In this section, we recall and introduce the basic definitions needed in this work.

Definition (2.1):

Let (X, τ) be a topological space and let T be an operator associated with τ, (X, T) is called an operator topological space, [2].

Let A ⊆ X, we say that A is T-semi open if there exists an open set U ∈ τ, such that:

U ⊆ A ⊆ T(U)

The complement of a T-semi open set is called a T-semi-closed set.

Remarks (2.2):

(i) If T is the closure operator (T(A) = cl(A)), the above definition agrees with the definition of semi-open set which is given by Levin in [1].

(ii) If T is the identity operator (T(A) = A) the definition of T-semi-open set agrees with the definition of open set.

(iii) For any operator T, each open set is T-semi-open set.

Example

Let (X, τ, T) be an operator topological space, where T is the closure operator and T – semi open set will be the usual semi – open set (the usual semi open set

G ⊆ W ⊆ G = cl(G) and T–semi open set G ⊆ W ⊆ T(G)

Consider X = ( IR, tu) and A= [ 0,1) is the T – semi open set ( T is the

*College of Education, Al-Mustansiryah University/Baghdad
** College of Science, Al-Kufa University/ Al-Najaf Ashraf

https://doi.org/10.30684/etj.28.19.12
2412-0758/University of Technology-Iraq, Baghdad, Iraq
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Before we state the next proposition, we recall the following definition:

**Definition (2.3), [2]:**

Let \((X, \tau, T)\) be an operator topological space. We say that \(T\) is a monotone operator if for every pair of open sets \(U\) and \(V\), such that \(U \subset V\), we have that \(T(U) \subset T(V)\).

The closure operator is a monotone operator.

**Proposition (2.4):**

Let \((X, \tau, T)\) be an operator topological space, where \(T\) is a monotone operator. Let \(A \subseteq X\), \(A\) is \(T\)-semiopen set if and only if \(A \subseteq T(\text{int}(A))\).

**Proof:**

Suppose that \(A \subseteq T(\text{int}(A))\)

Notice that:

\[ \text{int}(A) \subseteq A \subseteq T(\text{int}(A)) \]

which means that \(A\) is \(T\)-semiopen set

Now, suppose that \(A\) is \(T\)-semiopen set

Then there exists an open set \(U\), such that \(U \subseteq A \subseteq T(U)\)

Now, \(U \subseteq A \rightarrow U \subseteq \text{int}(A)\) [where \(\text{int}(U) \subseteq \text{int}(A)\) but \(U\) is open set, \(U= \text{int}(U)\)]

So, \(T(U) \subseteq T(\text{int}(A))\), since \(T\) is a monotone operator

So, \(A \subseteq T(\text{int}(A))\).

**Remark (2.5):**

If \(T\) is the closure operator, then \(A\) is \(T\)-semiopen set if and only if \(A \subseteq \text{cl}(\text{int}(A))\), and this agrees with the equivalence given by Levin in [1].

**Proposition (2.6):**

Let \((X, \tau, T)\) be an operator topological space, where \(T\) is the closure operator and let \(S \subseteq X\), then the union of all \(T\)-semi open sets contained in the set \(S\) is \(T\)-semi open, and it is denoted by \(T\)-sInt\((S)\).

We show this as follows:

\[ F = \{ w_\alpha: \alpha \in U \} \]

\(w_\alpha\) is the \(T\)-semi open

\[ G_\alpha \subseteq w_\alpha \subseteq T(G_\alpha) \]

\[ \bigcup G_\alpha \subseteq U \quad w_\alpha \subseteq \bigcup G_\alpha \subseteq T(G) \]

\[ \bigcup G_\alpha = T\left( \bigcup G_\alpha \right) = T(G) \]

**Remark (2.7):**

If \(T\) is a monotone operator, the intersection of all \(T\)-semi closed sets of \(X\) containing the set \(S\) is \(T\)-semi closed. It is called the \(T\)-semi closure of \(S\) and is denoted by \(T\)-scl\((S)\).

So, \(S\) is \(T\)-semi closed if and only if \(T\)-scl\((S)\) = \(S\).
3. Main Results:

In this section, several properties and characterizations of T-semi connected spaces are given.

**Definition (3.1):**

[See Willard [4] and Def (3.1)]

(i) Let $(X, \tau, T)$ be an operator topological space.
Let $\emptyset \neq A \subseteq X, \emptyset \neq B \subseteq X$
We say that $A$ and $B$ are T-semi separated if:

$$(T - \text{scl}(A)) \cap B = A \cap (T - \text{scl}(B)) = \emptyset$$

(ii) Let $W \subseteq X$, we say that $W$ is T-semi disconnected if $W$ can be expressed as the union of two T-semi separated sets.

(iii) Let $W \subseteq X$, we say that $W$ is T-semi connected if $W$ is not T-semi disconnected, that is, if $W$ cannot be expressed as the union of two T-semi separated sets.

(iv) $(X, \tau, T)$ is said to be T-semi connected if and only if $X$ is T-semi connected.

**Remark (3.2):**

Let $(X, \tau, T)$ be an operator topological space, where $T$ is the identity operator (In this case we write $(X, \tau)$ instead of $(X, \tau, T)$) let $A \subseteq X$, $B \subseteq X$. The definition of T-semi separated sets agrees with the definition of separated sets in the usual sense

$$\text{cl}(A) \cap B = A \cap \text{cl}(B) = \emptyset$$

and therefore the definition of T-semi connected set generalizes the definition of connected set.

**Example:**

Let $(X, \tau, T)$ be an operator topological space consider $X = (\mathbb{R}, \tau, t_u)$ and

$Y = [0,1] \cup [2,3]$ then $Y$ is not connected but $Y$ is T-semi connected (where $T$ is the closure operator).

**Theorem (3.3):**

Let $(X, \tau, T)$ be an operator topological space. Then $X$ is T-semi connected if and only if the only subsets of $X$ that are both T-semi open and T-semi closed in $X$ are the empty set and $X$ itself.

**Proof:**

Let $A \subseteq X$ be a non-empty proper subset of $X$, which is both T-semi open and T-semi closed in $X$, then the sets $U = A$ and $V = A^c$ constitute a T-semi separation of $X$. So $X$ will be T-semi disconnected.

Conversely, if $U$ and $V$ form T-semi separation of $X$ and $X = U \cup V$, then $U$ is a non-empty and different from $X$, since:

$$U \cap V \subseteq U \cap (T - \text{scl}(V)) = (T - \text{scl}(U)) \cap V = \emptyset$$

We obtain that both sets $U$ and $V$ are T-semi open and T-semi closed.

**Theorem (3.4):**

Let $(X, \tau, T)$ be an operator topological space, if $A \subseteq X$ is T-semi connected and $A \subseteq C \cup D$, where $C$ and $D$ are T-semi separated sets, then either $A \subseteq C$ or $A \subseteq D$.

**Proof:**

$$A = A \cap (C \cup D) = (A \cap C) \cup (A \cap D)$$

Since $C$ and $D$ are T-semi separated sets

$$C \cap (T - \text{scl}(D)) = \emptyset$$

so

$$(A \cap C) \cap (T - \text{scl}(A)) \cap (T - \text{scl}(D)) \subseteq C \cap (T - \text{scl}(D)) = \emptyset$$

So if both $A \cap C \neq \emptyset$ and $A \cap D \neq \emptyset$, then $A$ is T-semi disconnected. This shows that either $A \cap C = \emptyset$, or $A \cap D = \emptyset$.

So $A \subseteq C$, or $A \subseteq D$. ■
The proof of the following theorem is clear:

**Theorem (3.5):**

Let \((X, \tau, T)\) be an operator topological space, then the union \(E\) of any family \(\{C \mid i \in I\}\) of T-semi connected sets having a non-empty intersection is T-semi connected.

**Theorem (3.6):**

Let \((X, \tau, T)\) be an operator topological space, and let \(C \subseteq X, E \subseteq X\). If \(C\) is T-semi connected and \(C \subseteq (T-scl(E)) \subseteq T-scl(C)\) Then \((T-scl(E))\) is T-semi connected.

**Proof:**

Suppose that \(T-scl(E)\) is T-semi disconnected
So \(T-scl(E) = A \cup B\), where \(\emptyset \neq A, \emptyset \neq B\)
\[
A \cap (T-scl(B)) = B \cap (T-scl(A)) = \emptyset
\]
Now, \(C \subseteq A \cup B\) and \(C\) is T-semi connected, so:

\(C \subseteq A\) or \(C \subseteq B\)

Let us assume that \(C \subseteq A\)

Now, \(T-scl(C) \subseteq T-scl(A)\)

Therefore:

\[
T-scl(C) \cap B \subseteq T-scl(A) \cap B = \emptyset
\]

But \(B \subseteq T-scl(E) \subseteq T-scl(C)\)

So \(B = \emptyset\)

Hence, \(T-scl(E)\) is T-semi connected.

**Corollary (3.7):**

Let \((X, \tau, T)\) be an operator topological space, let \(C \subseteq X\). If \(C\) is T-semi connected, then \(T-scl(C)\) is also T-semi connected.

**Proof:**

\(C \subseteq T-scl(C) \subseteq T-scl(C)\)

So by theorem (3.6), we get that \(T-scl(C)\) is T-semi connected.

4. References:


