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Solving Linear and Non-Linear Eight Order Boundary Value Problems by Three Numerical Methods

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Abstract

Three numerical methods were implemented for solving the eight-order boundary value problems. These methods are Differential transformation method, Homotopy perturbation method, and Rung-Kutta of 4th Order method. Two physical problems from the literature were solved by these methods for comparing results. Solutions were presented in Tables and figures. The differential transformation method shows an effective numerical solution to linear boundary value problems. This considers an important contribution in solving boundary value problems by the differential transformation method.

Keywords: Boundary Value Problems, Ordinary Differential Equation, RK4, RK-Butcher

حل مسائل القيمة الحدية للمعادلات التفاضلية من المرتبة الثامنة باستخدام ثلاثة طرائق عددية مختلفة

الخلاصة

تم في هذا البحث حل مسائل القيمة الحدية للمعادلات التفاضلية من المرتبة الثامنة باستخدام ثلاثة طرائق عددية مختلفة. إن الطرائق العددية المستخدمة هي طريقة التحويل التفاضلي وطريقة الاضطراب الهوموتوبي و طريقة رانج كوتا من المرتبة الرابعة . تم تطبيق الطرائق الثلاثة لحل مسئلتين تطبيقيتين ومقارنة النتائج. تم عرض النتائج على شكل جداول ومخططات بيانية. لوحظ إن طريقة التحويل التفاضلي ذات كفاءة ودقة عالية في حل مسائل القيمة الحدية للمعادلات التفاضلية الخطية من المرتبة الثامنة .

1. Introduction

Let investigate the general eight-order boundary value problem of the type[1]:

$$y^{(viii)}(x)+f(x)y(x)=g(x), x\in[a,b] \quad (1.1)$$

With boundary conditions

$$y(a)=a_0, \quad y(b)=a_1,$$
$$y^{(2)}(a)=b_0, \quad y^{(2)}(b)=b_1,$$
$$y^{(4)}(a)=x_0, \quad y^{(4)}(b)=x_1,$$
$$y^{(6)}(a)=y_0, \quad y^{(6)}(b)=y_1 \quad (1.2)$$

Where $a_0,a_1,b_0,b_1,x_0,x_1,y_0,y_1$ are constants and $f(x),g(x)$: $(0,1]\times[0,\infty)$ $(0,1]$,continuous and $\frac{\partial f}{\partial y}$ exists and continuous and $\frac{\partial f}{\partial y}\geq 0$.

A class of characteristic-value problems of higher order is known to arise in hydrodynamic and hydromagnetic stability [2, 3]. When a layer of fluid is heated from below and is subject to the action of rotation, instability may set in as over stability [2, 4, 5]. This instability may be modeled by an eighth-order ordinary differential equation with appropriate boundary conditions [2, 5, 6]. For more

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discussion about the eighth-order boundary value problems, see [2–4, 6–8] and the references therein. The literature of numerical analysis contains little on the solution of the eighth-order boundary value problems [7]. Research in this direction may be considered in its early stages. Theorems which list the conditions for the existence and uniqueness of solutions of such problems are contained in a comprehensive survey by Agarwal [9].

In this paper we apply three numerical methods to solve Boundary Value Problems (BVPs), as solving Eq. (1.1) under the conditions (1.2). These methods are Differential transformation method [10], Homotopy perturbation method [11], and Rung-Kutta (RK4) [12]. Special program is designed to apply these methods. Two physical problems were studied before from [1] are solved by these methods. Special Computer programs are written in order to implement methods algorithms. Results are presented by tables and figures to compare errors and solutions for these methods.

2. Rung-Kutta 4th Order Method [12]:

Consider the boundary value problem in Eq.(1.1) under the condition in Eq.(1.2) , it will be transform to eight first-order equations by assuming:

$$\frac{dy}{dx} = p_1 \qquad (2.1)$$

$$\frac{dp_1}{dx} = p_2 \qquad (2.2)$$

$$\frac{dp_2}{dx} = p_3 \qquad (2.3)$$

$$\frac{dp_3}{dx} = p_4 \qquad (2.4)$$

$$\frac{dp_4}{dx} = p_5 \qquad (2.5)$$

$$\frac{dp_5}{dx} = p_6 \qquad (2.6)$$

$$\frac{dp_6}{dx} = p_7 \qquad (2.7)$$

Then Eqs.(1.1) becomes:

$$\frac{dp_7}{dx} = -f(x)y(x) + g(x) \qquad (2.8)$$

The solution $y(x)$ for this problem from $x_0 = 0$ to $x_n = 1$ after assuming h and n where:

$$h = \frac{x_n - x_0}{n} \qquad (2.9)$$

The set of 1st order (ODEs) from (2.1) to (2.8) are solved together from the following:

$$y^{n+1} = y^n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad (2.10)$$

$$p_1^{n+1} = p_1^n + \frac{1}{6}(L_1 + 2L_2 + 2L_3 + L_4) \qquad (2.11)$$

$$p_2^{n+1} = p_2^n + \frac{1}{6}(M_1 + 2M_2 + 2M_3 + M_4) \qquad (2.12)$$

$$p_3^{n+1} = p_3^n + \frac{1}{6}(N_1 + 2N_2 + 2N_3 + N_4) \qquad (2.13)$$

$$p_4^{n+1} = p_4^n + \frac{1}{6}(R_1 + 2R_2 + 2R_3 + R_4) \qquad (2.14)$$

$$p_5^{n+1} = p_5^n + \frac{1}{6}(S_1 + 2S_2 + 2S_3 + S_4) \quad (2.15)$$

$$p_6^{n+1} = p_6^n + \frac{1}{6}(T_1 + 2T_2 + 2T_3 + T_4) \quad (2.16)$$

$$p_7^{n+1} = p_7^n + \frac{1}{6}(U_1 + 2U_2 + 2U_3 + U_4) \quad (2.17)$$

Constants:
$k_1,k_2,k_3,k_4,L_1,L_2,L_3,L_4,M_1,M_2,M_3,M_4,N_1,N_2,N_3,N_4,R_1,R_2,R_3,R_4,S_1,S_2,S_3,S_4,T_1,T_2,T_3,T_4,U_1,U_2,U_3,U_4$

in Eqs.(2.10) to (2.17) are calculated from :

$$k_1 = h.fnf1(x, y, p_1, p_2, p_3, p_4, p_5, p_6, p_7) \quad (2.18)$$

$$L_1 = h.fnf2(x, y, p_1, p_2, p_3, p_4, p_5, p_6, p_7) \quad (2.19)$$

$$M_1 = h.fnf3(x, y, p_1, p_2, p_3, p_4, p_5, p_6, p_7) \quad (2.20)$$

$$N_1 = h.fnf4(x, y, p_1, p_2, p_3, p_4, p_5, p_6, p_7) \quad (2.21)$$

$$R_1 = h.fnf5(x, y, p_1, p_2, p_3, p_4, p_5, p_6, p_7) \quad (2.22)$$

$$S_1 = h.fnf6(x, y, p_1, p_2, p_3, p_4, p_5, p_6, p_7) \quad (2.23)$$

$$T_1 = h.fnf7(x, y, p_1, p_2, p_3, p_4, p_5, p_6, p_7) \quad (2.24)$$

$$U_1 = h.fnf8(x, y, p_1, p_2, p_3, p_4, p_5, p_6, p_7) \quad (2.25)$$

$$k_2 = h.fnf1(x+h/2, y+k_1/2, p_1 + L_1/2, p_2 + M_1/2, p_3 + N_1/2, p_4 + R_1/2, p_5 + S_1/2, p_6 + T_1/2, p_7 + U_1/2) \quad (2.26)$$

$$L_2 = h.fnf2(x+h/2, y+k_1/2, p_1 + L_1/2, p_2 + M_1/2, p_3 + N_1/2, p_4 + R_1/2, p_5 + S_1/2, p_6 + T_1/2, p_7 + U_1/2) \quad (2.27)$$

$$M_2 = h.fnf3(x+h/2, y+k_1/2, p_1 + L_1/2, p_2 + M_1/2, p_3 + N_1/2, p_4 + R_1/2, p_5 + S_1/2, p_6 + T_1/2, p_7 + U_1/2) \quad (2.28)$$

$$N_2 = h.fnf4(x+h/2, y+k_1/2, p_1 + L_1/2, p_2 + M_1/2, p_3 + N_1/2, p_4 + R_1/2, p_5 + S_1/2, p_6 + T_1/2, p_7 + U_1/2) \quad (2.29)$$

$$R_2 = h.fnf5(x+h/2, y+k_1/2, p_1 + L_1/2, p_2 + M_1/2, p_3 + N_1/2, p_4 + R_1/2, p_5 + S_1/2, p_6 + T_1/2, p_7 + U_1/2) \quad (2.30)$$

$$S_2 = h.fnf6(x+h/2, y+k_1/2, p_1 + L_1/2, p_2 + M_1/2, p_3 + N_1/2, p_4 + R_1/2, p_5 + S_1/2, p_6 + T_1/2, p_7 + U_1/2) \quad (2.31)$$

$$T_2 = h.fnf7(x+h/2, y+k_1/2, p_1 + L_1/2, p_2 + M_1/2, p_3 + N_1/2, p_4 + R_1/2, p_5 + S_1/2, p_6 + T_1/2, p_7 + U_1/2) \quad (2.32)$$

$$U_2 = h.fnf8(x+h/2, y+k_1/2, p_1 + L_1/2, p_2 + M_1/2, p_3 + N_1/2, p_4 + R_1/2, p_5 + S_1/2, p_6 + T_1/2, p_7 + U_1/2) \quad (2.33)$$

$$k_3 = h.fnf1(x+h/2, y+k_2/2, p_1 + L_2/2, p_2 + M_2/2, p_3 + N_2/2, p_4 + R_2/2, p_5 + S_2/2, p_6 + T_2/2, p_7 + U_2/2) \quad (2.34)$$

$$L_3 = h.fnf2(x+h/2, y+k_2/2, p_1 + L_2/2, p_2 + M_2/2, p_3 + N_2/2, p_4 + R_2/2, p_5 + S_2/2, p_6 + T_2/2, p_7 + U_2/2) \quad (2.35)$$

$$M_3 = h.fnf3(x+h/2, y+k_2/2, p_1 + L_2/2, p_2 + M_2/2, p_3 + N_2/2, p_4 + R_2/2, p_5 + S_2/2, p_6 + T_2/2, p_7 + U_2/2) \quad (2.36)$$

$$N_3 = h.fnf4(x+h/2, y+k_2/2, p_1 + L_2/2, p_2 + M_2/2, p_3 + N_2/2, p_4 + R_2/2, p_5 + S_2/2, p_6 + T_2/2, p_7 + U_2/2) \quad (2.37)$$

$$R_3 = h.fnf5(x+h/2, y+k_2/2, p_1 + L_2/2, p_2 + M_2/2, p_3 + N_2/2, p_4 + R_2/2, p_5 + S_2/2, p_6 + T_2/2, p_7 + U_2/2) \quad (2.38)$$

$$S_3 = h.fnf6(x+h/2, y+k_2/2, p_1 + L_2/2, p_2 + M_2/2, p_3 + N_2/2, p_4 + R_2/2, p_5 + S_2/2, p_6 + T_2/2, p_7 + U_2/2) \quad (2.39)$$

$$T_3 = h.fnf7(x+h/2, y+k_2/2, p_1 + L_2/2, p_2 + M_2/2, p_3 + N_2/2, p_4 + R_2/2, p_5 + S_2/2, p_6 + T_2/2, p_7 + U_2/2) \quad (2.40)$$

$$U_3 = h.fnf8(x+h/2, y+k_2/2, p_1 + L_2/2, p_2 + M_2/2, p_3 + N_2/2, p_4 + R_2/2, p_5 + S_2/2, p_6 + T_2/2, p_7 + U_2/2) \quad (2.41)$$

(2.42)

$$L_4 = h.fnf2(x+h, y+k_3, p_1 + L_3, p_2 + M_3, p_3 + N_3, p_4 + R_3, p_5 + S_3, p_6 + T_3, p_7 + U_3) \quad (2.43)$$

$$M_4 = h.fnf3(x+h, y+k_3, p_1 + L_3, p_2 + M_3, p_3 + N_3, p_4 + R_3, p_5 + S_3, p_6 + T_3, p_7 + U_3) \quad (2.44)$$

$$N_4 = h.fnf4(x+h, y+k_3, p_1 + L_3, p_2 + M_3, p_3 + N_3, p_4 + R_3, p_5 + S_3, p_6 + T_3, p_7 + U_3) \quad (2.45)$$

$$R_4 = h.fnf5(x+h, y+k_3, p_1 + L_3, p_2 + M_3, p_3 + N_3, p_4 + R_3, p_5 + S_3, p_6 + T_3, p_7 + U_3) \quad (2.46)$$

$$S_4 = h.fnf6(x+h, y+k_3, p_1 + L_3, p_2 + M_3, p_3 + N_3, p_4 + R_3, p_5 + S_3, p_6 + T_3, p_7 + U_3) \quad (2.47)$$

$$T_4 = h.fnf7(x+h, y+k_3, p_1 + L_3, p_2 + M_3, p_3 + N_3, p_4 + R_3, p_5 + S_3, p_6 + T_3, p_7 + U_3) \quad (2.48)$$

$$U_4 = h.fnf8(x+h, y+k_3, p_1 + L_3, p_2 + M_3, p_3 + N_3, p_4 + R_3, p_5 + S_3, p_6 + T_3, p_7 + U_3) \quad (2.49)$$

3. Differential Transformation Method DTM [10]:

The differential transformation of the k-th derivatives of function $y(x)$ is defined as follows [10]:

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y}{dx^k}\right]_{x=x_0} \quad (3.1)$$

and $y(x)$ is the differential inverse transformation of $Y(k)$ defined as follows:

$$y(x)=\sum_{k=0}^{\infty}Y(k).(x-x_0)^k \qquad (3.2)$$

for finite series of $k = N$, Eq.(3.2) can be written as:

$$y(x)=\sum_{k=0}^{N}Y(k).(x-x_0)^k \qquad (3.3)$$

The following theorems that can be deduced from Eqs.(3.1) and (3.3) are given below [10]:

Theorem 3.1.
If $y(x)=g(x)\pm h(x)$,then $Y(k)=G(k)\pm H(k)$.

Theorem 3.2. If $y(x)=a.g(x)$,then $Y(k)=a.G(k)$.

Theorem 3.3. If $y(x)=\frac{dg(x)}{dx}$,then $Y(k)=(k+1).G(k+1)$.

Theorem 3.4. If $y(x)=\frac{d^m g(x)}{dx^m}$,then $Y(k)=((k+m)!/k!).G(k+m)$.

Theorem 3.5. If $y(x)=g(x).h(x)$,then $Y(k)=\sum_{l=0}^{k}G(l)H(k-l)$.

Theorem 3.6. If $y(x)=x^m$,then $Y(k)=d(k-m)=\begin{cases}1 & if \quad k=m\\ 0 & if \quad k\neq m\end{cases}$

Theorem 3.7. If $y(x)=\exp(a.x)$,then $Y(k)=a^k/k!$.

Theorem 3.8. If $y(x)=\sin(a.x+l)$,then $Y(k)=(a^k/k!)\sin(kp/2+l)$.

Theorem 3.9.
If $y(x)=\cos(a.x+l)$,then $Y(k)=(a^k/k!)\cos(kp/2+l)$.

Theorem 3.10. If $y(x)=\exp(u(x))$,

then:

$$Y(k)=\sum_{k=0}^{\infty}\sum_{n=0}^{r}\frac{1}{n!}\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdot\cdot\sum_{k2=0}^{k3}\sum_{k1=0}^{k2}U(k_1)U(k_2-k_1)\ldots\times$$

$$U(k_{n-1}-k_{n-2})U(k-k_{n-1})(x-k_0)^k$$

4. Homotopy Perturbation Method HPM [11] :

Consider the following system of the integral equations [11]:

$$F(t)=G(t)+l\int_0^t K(t,s)F(s)ds \qquad (4.1)$$

$$F(t)=(f_1(t),\ f_2(t),\ldots\ldots,\ f_n(t))^T$$

$$G(t)=(g_1(t),\ g_2(t),\ldots\ldots,\ g_n(t))^T \qquad (4.2)$$

$$K(t,s)=[K_{ij}(t,s)]$$

$$i=1,2,3,\ldots..,n:\ j=1,2,3,\ldots\ldots,n$$

Let

$$L(y)=0 \qquad (4.3)$$

Where L an integral or differential operator, and then a convex homotopy is $H(y,p)$ defines by:

$$H(y,p)=(1-p)F(y)+pL(y) \qquad (4.4)$$

Where $F(y)$ is a functional operator with known solution v_0, which can be obtained easily. It is clear that

$$H(y,p)=0 \qquad (4.5)$$

From which we have

$$H(y,0)=F(y),$$

and $H(y,1)=L(y)$

This shows that $H(y,p)$ continuously traces an implicitly defined curve from a starting point $H(v_0,0)$ to a solution $H(f,1)$. The embedding parameter increases monotonically from zero to unit at the problem $F(y)=0$ is continuously deforms the original problem $L(y)=0$. The embedding parameter can be considered as an expanding parameter [11].

The homotopy perturbation method uses the homotopy parameter p as an expanding parameter to obtain:

$$y=\sum_{i=0}^{\infty} p^i y_i = y_0 + y_1 p + y_2 p^2 + y_3 p^3 + y_4 p^4 + y_5 p^5 + ... \quad (4.6)$$

If $p \to 1$ then (4.5) corresponds to (4.3) and becomes the approximate solution of the form

$$f=\lim_{p\to 1} y = \sum_{i=0}^{\infty} y_i \quad (4.7)$$

It is well known that the series (4.7) is convergent for most of the cases and also the rate of convergent is dependent on $L(y)$, see [13]. We assume that problem (4.1) has a unique solution.

Consider the ith equation of (4.1), take

$$f_1(t)=\sum_{i=0}^{\infty} p^i y_i \ , \quad f_2(t)=\sum_{i=0}^{\infty} p^i v_i \ ,$$
$$f_3(t)=\sum_{i=0}^{\infty} p^i w_i \ , \ ...... \quad (4.8)$$

The comparison of like powers of p gives solution of various orders.

5. Numerical Examples

Some of the physical problems are solved to assign the effectiveness and accuracy of the three methods [1]. Results are presented in tables and figures. Computer programs were written to implement procedures of the three methods.

5.1 Example 1 Consider the following eight-order non-linear boundary value problem [1]:

$$\frac{d^{(8)}y}{dx^8} = e^{-x}.y^2(x) \qquad 0<x<1 \quad (5.1.1)$$

Subject to the boundary conditions

$$y(0)=y^{//}(0)=y^{(iv)}(0)=y^{(vi)}(0)=1 \ ,$$
$$y(1)=y^{//}(1)=y^{(iv)}(1)=y^{(vi)}(1)=e \quad (5.1.2)$$

The exact solution is given by

$$y(x)=e^{x} \quad (5.1.3)$$

This problem was studied by [1] by applying Variation iteration decomposition method (VIDM).

5.1.1 Differential Transformation Method

By applying the Differential Transformation Method using theorems 3.1, 3.2, 3.4, and 3.5 to Eq. (5.1.1) the recurrence relation can be evaluated as follows:

$$Y(k+8)=\frac{k!}{(k+8)!}[\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\frac{(-1)^{(k-k_2)}}{(k-k_2)!} Y(k_1)Y(k_2-k_1)] \quad (5.1.4)$$

The boundary conditions in Eq.(5.1.2) can be transformed at $x_0=0$ as:

$$Y(0)=1, Y(2)=1/2, Y(4)=1/24, Y(6)=1/720 \quad (5.1.5)$$

Then:

$$y(x)=1+Y(1)x+\frac{1}{2}x^2+Y(3)x^3+\frac{1}{24}x^4$$
$$+Y(5)x^5+\frac{1}{720}x^6+Y(7)x^7+\frac{1}{8!}x^8+$$
$$\frac{1}{9!}(-1+2Y(1))x^9+\frac{1}{10!}(3-4Y(1))x^{10}+$$
$$\frac{1}{11!}(-7+12Y(1)+12Y(3))x^{11}+$$
$$\frac{1}{12!}(21-8Y(1))x^{12}+O(x^{13})$$

$$\sum_{k=0}^{N} Y(k)=e, \quad (5.1.6)$$

$$\sum_{k=0}^{N} (k+2)!Y(k+2)/k!=e \quad (5.1.7)$$

$$\sum_{k=0}^{N} (k+4)!Y(k+4)/k!=e \quad (5.1.8)$$

$$\sum_{k=0}^{N} (k+6)!Y(k+6)/k!=e \quad (5.1.9)$$

For N=14 and by using the recurrence relations in Eq. (5.1.4) and the transformed boundary conditions in Eqs. (5.1.6) to (5.1.9), the following set of equations obtained

$$:\begin{bmatrix}1 & 1 & 1 & 1\\ .00032 & 6.0000 & 20 & 42\\ .01269 & .00238 & 120 & 840\\ .22222 & .10000 & 0 & 5040\end{bmatrix}.\begin{bmatrix}Y(1)\\Y(3)\\Y(5)\\Y(7)\end{bmatrix}=\begin{bmatrix}1.175197\\1.176752\\1.181649\\1.238888\end{bmatrix} \quad (5.1.10)$$

Solving Eq. (5.1.10) then:

$Y(1)=0.9987$, $Y(3)=0.1668$,

$Y(5)=0.0082$, $Y(7)=0.00021$

Then:

$$y(x)=1+0.9987x+0.5x^2+0.1668x^3+$$
$$\frac{1}{24}x^4+0.0082x^5+\frac{1}{720}x^6+0.00021x^7+$$
$$\frac{1}{40320}x^8+2.75\times10^{-6}\ x^9-2.7\times10^{-7}\ x^{10}$$
$$+2.5\times10^{-8}\ x^{11}-2.1\times10^{-9}\ x^{12}+O(x^{13})$$
(5.1.11)

5.1.2 Homotopy Perturbation Method

To solve the BVPs in Eq. (5.1.1) by applying the Homotopy Perturbation Method, we can rewrite the eight-order boundary value problem as a system of eight differential equations:

$$\frac{dy(x)}{dx}=a(x), \quad \frac{da(x)}{dx}=b(x),$$
$$\frac{db(x)}{dx}=e(x)$$
$$\frac{de(x)}{dx}=f(x), \quad \frac{df(x)}{dx}=g(x),$$
$$\frac{dg(x)}{dx}=h(x) \quad (5.1.12)$$
$$\frac{dh(x)}{dx}=z(x),$$
$$\frac{dz(x)}{dx}=e^{-x}.y^2(x)$$

This can be written as a system of integral equations:

$$y(x)=1+\int_0^x a(t).dt$$
$$a(x)=A+\int_0^x b(t).dt$$
$$b(x)=1+\int_0^x e(t).dt$$
$$e(x)=B+\int_0^x f(t).dt$$

$$f(x) = 1 + \int_0^x g(t).dt$$
$$g(x) = C + \int_0^x h(t).dt$$
$$h(x) = 1 + \int_0^x z(t).dt \qquad (5.1.13)$$
$$z(x) = D + \int_0^x e^{-t}.y^2(t)dt$$

Using (4.4) and (4.6) for (5.1.13) we have:

$$y_0 + py_1 + p^2y_2 + p^3y_3 + ...= 1 + p\int_0^x (a_0 + pa_1 + p^2a_2 + p^3a_3 + ...).dt$$

$$a_0 + pa_1 + p^2a_2 + p^3a_3 + ...= A + p\int_0^x (b_0 + pb_1 + p^2b_2 + p^3b_3 + ...).dt$$

$$b_0 + pb_1 + p^2b_2 + p^3b_3 + ...= 1 + p\int_0^x (e_0 + pe_1 + p^2e_2 + p^3e_3 + ...).dt$$

$$e_0 + pe_1 + p^2e_2 + p^3e_3 + ...= B + p\int_0^x (f_0 + pf_1 + p^2f_2 + p^3f_3 + ...).dt$$

$$f_0 + pf_1 + p^2f_2 + p^3f_3 + ...= 1 + p\int_0^x (g_0 + pg_1 + p^2g_2 + p^3g_3 + ...).dt$$

$$g_0 + pg_1 + p^2g_2 + p^3g_3 + ...= C + p\int_0^x (h_0 + ph_1 + p^2h_2 + p^3h_3 + ...).dt$$

$$h_0 + ph_1 + p^2h_2 + p^3h_3 + ...= 1 + p\int_0^x (z_0 + pz_1 + p^2z_2 + p^3z_3 + ...).dt$$

$$z_0 + pz_1 + p^2z_2 + p^3z_3 + ...= D + p\int_0^x e^{-t}.y^2(t)dt \qquad (5.1.14)$$

Comparing the coefficient of like powers of p, we have

$$p^{(0)}: \begin{cases} y_0 = 1 \\ a_0 = A \\ b_0 = 1 \\ e_0 = B \\ f_0 = 1 \\ g_0 = C \\ h_0 = 1 \\ z_0 = D \end{cases}, \quad p^{(1)}: \begin{cases} y_1 = 1 + Ax \\ a_1 = A + x \\ b_1 = 1 + Bx \\ e_1 = B + x \\ f_1 = 1 + Cx \\ g_1 = C + x \\ h_1 = 1 + Dx \\ z_1 = 1 - e^{-x} \end{cases},$$

$$p^{(2)}: \begin{cases} y_2 = 1 + Ax + x^2/2 \\ a_2 = A + x + Bx^2/2 \\ b_2 = 1 + Bx + x^2/2 \\ e_2 = B + x + Cx^2/2 \\ f_2 = 1 + Cx + x^2/2 \\ g_2 = C + x + Dx^2/2 \\ h_2 = 1 + Dx - 1 + x + e^{-x} \\ z_2 = 1 - e^{-x} + A^2(2 + 2x + \\ x^2 - 3e^{-x} - x^2e^{-x}) \end{cases}$$

$$p^{(3)}:\begin{cases} y_3 = 1 + Ax + x^2/2 + Bx^3/3! \\ a_3 = A + x + Bx^2/2 + x^3/3! \\ b_3 = 1 + Bx + x^2/2 + Cx^3/3! \\ e_3 = B + x + Cx^2/2 + x^3/3! \\ f_3 = 1 + Cx + x^2/2 + Dx^3/3! \\ g_3 = C + x + Dx^2/2 + 1 - x + \\ \frac{1}{2}x^2 - e^{-x} \\ h_3 = 1 + Dx - 1 + x + e^{-x} \\ z_3 = 1 - e^{-x} + A^2(2 + 2x + x^2 \\ -3e^{-x} - x^2e^{-x}) \end{cases}$$

(5.1.15)

Adding all the terms, (5.1.15) gives:

$$y(x) = 1 + Ax - \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 - \frac{1}{4!}x^4 + \frac{1}{5!}Cx^5 - \frac{1}{6!}x^6 + \frac{1}{7!}Dx^7 + \frac{1}{8!}x^8 + \frac{1}{9!}(2A-1)x^9 + \frac{1}{10!}(3-4A)x^{10} + \frac{1}{11!}(-7+6A+2B)x^{11} + \frac{1}{12!}(15-8A-8B)x^{12} + O(x^{13})$$

(5.1.16)

Using the boundary conditions at x=1, then lead to the following system of equations

$$\begin{bmatrix} 1 & \frac{1}{6} & \frac{1}{120} & \frac{1}{5040} \\ .0003 & 1 & \frac{1}{6} & \frac{1}{120} \\ .0121 & \frac{1}{5040} & 1 & \frac{1}{6} \\ .2022 & \frac{1}{180} & 0 & \frac{37}{180} \end{bmatrix} . \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 1.175203 \\ 1.175365 \\ 1.181798 \\ 1.297440 \end{bmatrix}$$

(5.1.16)

The solution of (5.1.16) gives:

$A = 0.999870,\quad B = 1.001257,$

$C = 0.988439,\quad D = 1.086357$

(5.1.17)

The solution is given as:

$$y(x) = 1 + 0.99870x + \frac{1}{2}x^2 + 0.166876x^3 + \frac{1}{24}x^4 + 0.008237x^5 + \frac{1}{720}x^6 + 0.0002155x^7 + \frac{1}{40320}x^8 + 2.755\times10^{-6}\ x^9 - 2.75\times10^{-7}\ x^{10} + 2.51\times10^{-8}\ x^{11} - 2.1\times10^{-9}\ x^{12} + O(x^{13})$$

(5.1.18)

5.1.3 Rung Kutta Method RK4

To solve the Boundary Value Problem in Eq. (5.1.1) by applying RK4 Method, we can rewrite the eight-order boundary value problem as a system of eight first order differential equations. By using algorithm presented above for RK4 method, we can get the solution of Eq. (5.1.1).

Solutions of the three methods are presented in Table (1), while errors are summarized in Table (2). Figure (1) presents solutions of problem in Example (1) by the different methods. Figure (2) shows the comparison between absolute errors of methods for solving the problem in Example (1).

5.2 Example 2 Consider the following eight-order linear boundary value problem [1]:

$$\frac{d^{(8)}y}{dx^8} = -8xe^{x} + y(x) \qquad 0 < x < 1$$

(5.2.1)

Subject to the boundary conditions

$$y(0) = 1, y^{//}(0) = -1, y^{(iv)}(0) = -3, y^{(vi)}(0) = -5$$

$$y(1) = 0, y^{//}(1) = -2e, y^{(iv)}(1) = -4e,$$
$$y^{(vi)}(1) = -6e$$

(5.2.2)

The exact solution is given by

$$y(x)=(1-x)e^{x} \qquad (5.2.3)$$

This problem was studied by [1] by applying Variational iteration decomposition method (VIDM).

5.2.1Differential Transformation Method

By applying the Differential Transformation Method using theorems 3.1, 3.2, 3.4, and 3.5 to Eq. (5.2.1) the recurrence relation can be evaluated as follows:

$$Y(k+8)=\frac{k!}{(k+8)!}[Y(k)-8\sum_{l=0}^{k}\frac{d(k-l)}{(k)!}] \qquad (5.2.4)$$

The boundary conditions in Eq.(5.2.2) can be transformed at $x_0=0$ as:

$$Y(0)=1, Y(2)=-1/2, Y(4)=-1/8,$$
$$Y(6)=-1/144 \qquad (5.2.5)$$

Then:

$$y(x)=1+Y(1)x-\frac{1}{2}x^2+Y(3)x^3-\frac{1}{8}x^4+Y(5)x^5-\frac{1}{144}x^6+Y(7)x^7-\frac{7}{8!}x^8+\frac{1}{9!}(-8+Y(1))x^9-\frac{9}{10!}x^{10}+\frac{1}{11!}(-8+6Y(3))x^{11}-\frac{11}{12!}x^{12}+O(x^{13}) \qquad (5.2.6)$$

$$\sum_{k=0}^{N} Y(k)=0, \qquad (5.2.7)$$

$$\sum_{k=0}^{N} (k+2)!Y(k+2)/k!=-2e \qquad (5.2.8)$$

$$\sum_{k=0}^{N} (k+4)!Y(k+4)/k!=-4e \qquad (5.2.9)$$

$$\sum_{k=0}^{N} (k+6)!Y(k+6)/k!=-6e$$

(5.2.10)

For N=12 and by using the recurrence relations in Eq. (5.2.4) and the transformed boundary conditions in Eqs. (5.2.7) to (5.2.10), the following set of equations obtained:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{5040} & 6 & 20 & 42 \\ \frac{1}{120} & \frac{1}{840} & 120 & 840 \\ \frac{1}{6} & \frac{1}{20} & \frac{1}{42} & 5040 \end{bmatrix} \cdot \begin{bmatrix} Y(1) \\ Y(3) \\ Y(5) \\ Y(7) \end{bmatrix} = \begin{bmatrix} -0.367857193 \\ -2.716672281 \\ -5.000408233 \\ -6.017503471 \end{bmatrix} \qquad (5.2.11)$$

Solving Eq. (5.2.11) then:

$$Y(1)=0, Y(3)=-0.33333333,$$
$$Y(5)=-0.03333333, Y(7)=-0.0011904\ldots$$

Then:

$$y(x)=1-\frac{1}{2}x^2-0.3333333x^3-\frac{1}{24}x^4-0.0333333x^5-\frac{1}{144}x^6-0.00119047x^7-\frac{1}{5760}x^8-2.2045\times10^{-5}x^9-\frac{1}{403200}x^{10}-2.505\times10^{-7}x^{11}-\frac{1}{43545600}x^{12}+O(x^{13}) \qquad (5.2.12)$$

5.2.2 Homotopy Perturbation Method

To solve the BVPs in Eq. (5.2.1) by applying the Homotopy Perturbation Method, we can rewrite the eight-order boundary value problem as a system of eight differential equations:

$$\frac{dy(x)}{dx}=a(x)\ ,\qquad \frac{da(x)}{dx}=b(x),$$
$$\frac{db(x)}{dx}=e(x)$$
$$\frac{de(x)}{dx}=f(x)\ ,\qquad \frac{df(x)}{dx}=g(x),$$
$$\frac{dg(x)}{dx}=h(x)$$
(5.2.13)
$$\frac{dh(x)}{dx}=z(x)\ ,$$
$$\frac{dz(x)}{dx}=-8xe^{x}+y(x)$$

This can be written as a system of integral equations:

$$y(x)=1+\int_0^x a(t).dt$$
$$a(x)=A+\int_0^x b(t).dt$$
$$b(x)=-1+\int_0^x e(t).dt$$
$$e(x)=B+\int_0^x f(t).dt$$
$$f(x)=-3+\int_0^x g(t).dt$$
$$g(x)=C+\int_0^x h(t).dt$$
(5.2.14)
$$h(x)=-5+\int_0^x z(t).dt$$
$$z(x)=D+\int_0^x [-8te^{t}+y(t)]dt$$

Using (4.4) and (4.6) for (5.2.14) we have:

$$y_0+py_1+p^2y_2+p^3y_3+........=$$
$$1+p\int_0^x (a_0+pa_1+p^2a_2+p^3a_3+........).dt$$

$$a_0+pa_1+p^2a_2+p^3a_3+........=$$
$$A+p\int_0^x (b_0+pb_1+p^2b_2+p^3b_3+........).dt$$

$$b_0+pb_1+p^2b_2+p^3b_3+..........=$$
$$1+p\int_0^x (e_0+pe_1+p^2e_2+p^3e_3+........).dt$$

$$e_0+pe_1+p^2e_2+p^3e_3+.........=$$
$$B+p\int_0^x (f_0+pf_1+p^2f_2+p^3f_3+........).dt$$

$$f_0+pf_1+p^2f_2+p^3f_3+........=$$
$$1+p\int_0^x (g_0+pg_1+p^2g_2+p^3g_3+........).dt$$

$$g_0+pg_1+p^2g_2+p^3g_3+.......=$$
$$C+p\int_0^x (h_0+ph_1+p^2h_2+p^3h_3+........).dt$$

$$h_0+ph_1+p^2h_2+p^3h_3+..........=$$
$$1+p\int_0^x (z_0+pz_1+p^2z_2+p^3z_3+........).dt$$

$$z_0+pz_1+p^2z_2+p^3z_3+............=$$
$$D+p\int_0^x [-8te^{t}+y(t)]dt$$

(5.2.15)

Comparing the coefficient of like powers of p, we have

$$p^{(0)}:\begin{cases} y_0 = 1 \\ a_0 = A \\ b_0 = -1 \\ e_0 = B \\ f_0 = -3 \\ g_0 = C \\ h_0 = -5 \\ z_0 = D \end{cases}$$

$$, p^{(1)}:\begin{cases} y_1 = 1 + Ax \\ a_1 = A - x \\ b_1 = -1 + Bx \\ e_1 = B - 3x \\ f_1 = -3 + Cx \\ g_1 = C - 5x \\ h_1 = -5 + Dx \\ z_1 = D - 8 + x + 8e^x - 8xe^x \end{cases}$$

,

$$p^{(2)}:\begin{cases} y_2 = 1 + Ax - x^2/2 \\ a_2 = A - x + Bx^2/2 \\ b_2 = -1 + Bx - 3x^2/2 \\ e_2 = B - 3x + Cx^2/2 \\ f_2 = -3 + Cx - 5x^2/2 \\ g_2 = C - 5x + Dx^2/2 \\ h_2 = -5 + Dx - 16 + 8x + \frac{1}{2}x^2 + \\ 16e^x - 8xe^x \\ z_2 = D - 16 + x + 16e^x - 16xe^x + \\ \frac{1}{2}Ax^2 - \frac{1}{6}x^3 \end{cases}$$

$$p^{(3)}:\begin{cases} y_3 = 1 + Ax - x^2/2 + Bx^3/6 \\ a_3 = A - x + Bx^2/2 - x^3/2 \\ b_3 = -1 + Bx - 3x^2/2 + Cx^3/6 \\ e_3 = B - 3x + Cx^2/2 - 5x^3/6 \\ f_3 = -3 + Cx - 5x^2/2 + Dx^3/6 \\ g_3 = C - 5x + Dx^2/2 - 24 - 16x + \\ 4x^2 + x^3/6 + 24e^x - 8xe^x \\ h_3 = -5 + Dx - 32 + 16x + x^2/2 - 16x \\ + 4x^2 + Ax^3/6 + 32e^x - 16xe^x \\ z_3 = D - 24 + x + 24e^x - 24xe^x + \frac{1}{2}Ax^2 \\ -\frac{1}{6}x^3 \end{cases}$$

(5.2.16)

Adding all the terms, (5.2.16) gives:

$$y(x) = 1 + Ax - \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 - \frac{1}{8}x^4 + \frac{1}{5!}Cx^5 - \frac{1}{144}x^6 + \frac{1}{7!}Dx^7 - \frac{1}{5760}x^8 + \frac{1}{9!}(A-8)x^9 - \frac{9}{10!}x^{10} + \frac{1}{11!}(-8+6B)x^{11} - \frac{11}{12!}x^{12} + O(x^{13})$$

(5.2.17)

Using the boundary conditions at x=1, then lead to the following system of equations:

$$\begin{bmatrix} 1 & \frac{1}{6} & \frac{1}{120} & \frac{1}{5040} \\ \frac{1}{5040} & 1 & \frac{1}{6} & \frac{1}{120} \\ \frac{1}{120} & \frac{1}{5040} & 1 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{120} & \frac{1}{5040} & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} -0.3678571193 \\ -2.716672281 \\ -5.000408233 \\ -6.01750347 \end{bmatrix}$$
(5.2.18)

The solution of (5.1.18) gives

$A = 0, \quad B = -2.0000065,$

$C = -3.99994, \quad D = -6.00036$ (5.2.19)

The solution is given as:

$$y(x) = 1 - 0.5x^2 - 0.3333333\,x^3 - \frac{1}{8}x^4 - 0.03333333\,x^5 - \frac{1}{144}x^6 - 0,0011904\,x^7 - \frac{1}{5760}x^8 - 2.204 \times 10^{-5}x^9 - \frac{1}{403200}x^{10} - 2.505 \times 10^{-7}x^{11} - \frac{1}{43545600}x^{12} + O(x^{13})$$
(5.2.20)

5.2.3 Rung Kutta Method RK4

To solve the Boundary Value Problem in Eq. (5.2.1) by applying RK4 Method, we can rewrite the eight-order boundary value problem as a system of eight first order differential equations. By using algorithm presented above for RK4 method, we can get the solution of Eq. (5.2.1).

Solutions of the different methods are presented in Table (3),while errors are summarized in Table (4). Figure (3) presents solutions of problem in Example (2) by the different methods. Figure (4) shows the comparison between absolute errors of methods for solving the problem in Example (2).

Conclusions

In this paper, three numerical methods DTM, HPM, and RK4 were applied for finding the solution of eight-order two-point nonlinear and linear boundary value problems. It may be concluded that the differential transformation method shows an effective tool in finding the numerical solutions to linear boundary value problems.

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Table (1) Solutions of the problem in Example (1).

X	*Exact Solution*	DTM Solution	*HPM solution*	*RK4 solution*
0.1	1.105171	1.105041	1.105041	1.105171
0.2	1.221403	1.221144	1.221144	1.221403
0.3	1.349859	1.349472	1.349474	1.349858
0.4	1.491825	1.491312	1.491317	1.491824
0.5	1.648721	1.648084	1.648094	1.648721
0.6	1.822119	1.821357	1.821376	1.822118
0.7	2.013753	2.012867	2.012898	2.013752
0.8	2.225541	2.224528	2.224577	2.225539
0.9	2.459603	2.458457	2.458533	2.459601
1.0	2.718282	2.716993	2.717106	2.71828

Table (2) Errors of the three numerical methods of Example (1).

X	DTM *Error*	*HPM Error*	*RK4 Error*
0.1	1.299381E-04	1.298189E-04	1.192093 E-07
0.2	2.589226E-04	2.583265E-04	2.384186 E-07
0.3	3.867149E-04	3.848076E-04	4.768372 E-07
0.4	5.128384E-04	5.079508E-04	5.960464 E-07
0.5	6.372929E-04	6.273985E-04	7.152557 E-07
0.6	7.613897E-04	7.431507E-04	1.072884 E-06
0.7	8.857251E-04	8.549691E-04	1.192093 E-06
0.8	1.013279E-03	9.641647E-04	1.66893E -06
0.9	1.146078E-03	1.070738E-03	1.907349 E-06
1.0	1.289129E-03	1.176357E-03	2.384186 E-06

Table (3) Solutions of the problem in Example (2)

X	*Exact Solution*	DTM *Solution*	*HPM Solution*	*RK4 solution*
0.1	0.9946538	0.9946538	0.9946538	0.9946542
0.2	0.9771222	0.9771222	0.9771222	0.977123
0.3	0.9449012	0.9449012	0.9449012	0.9449025
0.4	0.8950948	0.8950948	0.8950948	0.8950969
0.5	0.8243606	0.8243606	0.8243606	0.8243641
0.6	0.7288475	0.7288475	0.7288475	0.7288545
0.7	0.6041257	0.6041257	0.6041257	0.604142
0.8	0.4451081	0.4451081	0.4451081	0.4451473
0.9	0.2459601	0.2459601	0.2459602	0.2460524
1.0	0.9946538	0.9946538	0.9946538	0.9946542

Table (4) Errors of the three numerical methods of Example (2)

X	DTM Error	HPM Error	RK4 Error
0.1	1.6E-10	1.0E-9	3.576279E-7
0.2	2.4E-10	1.3E-9	7.748604E-7
0.3	1.0E-9	1.6E-9	1.311302E-6
0.4	1.1E-9	1.7E-9	2.086163E-6
0.5	1.3E-9	1.8E-9	3.516674E-6
0.6	1.7E-9	2.3E-9	6.973743E-6
0.7	1.8E-9	2.1E-8	1.621246E-5
0.8	2.3E-9	5.9E-8	3.921986E-5
0.9	1.5E-8	9.8E-8	9.231269E-5

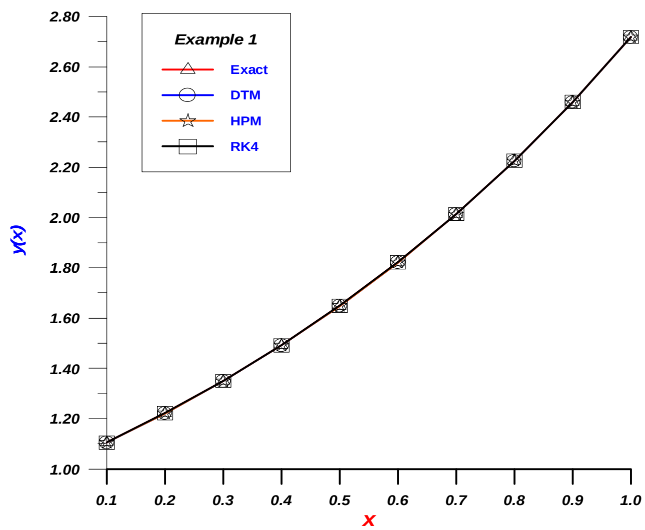

Figure (1) Solutions of problem in Example (1)

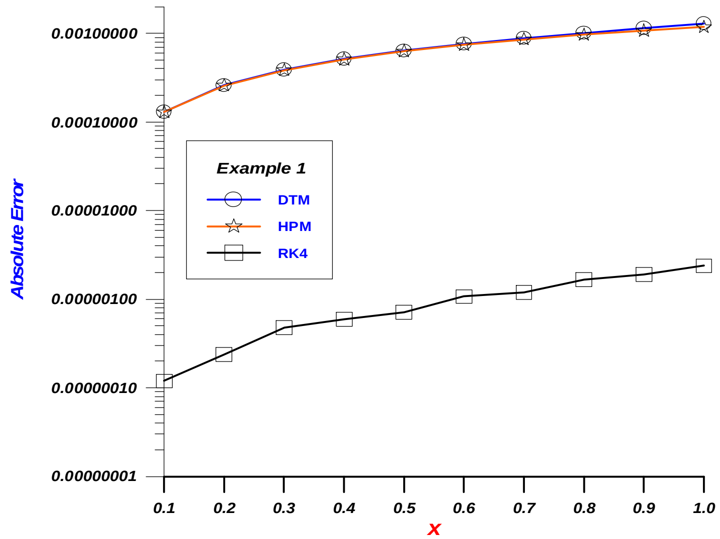

Figure (2) Errors for methods of Example (1)

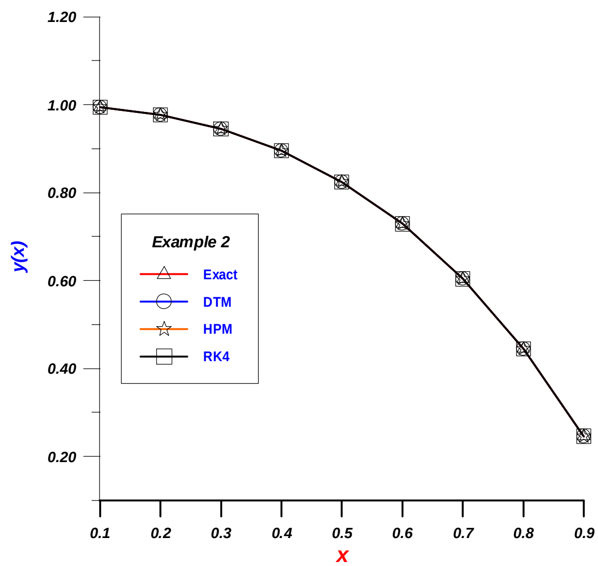

Figure (3) Solutions of problem in Example (2)

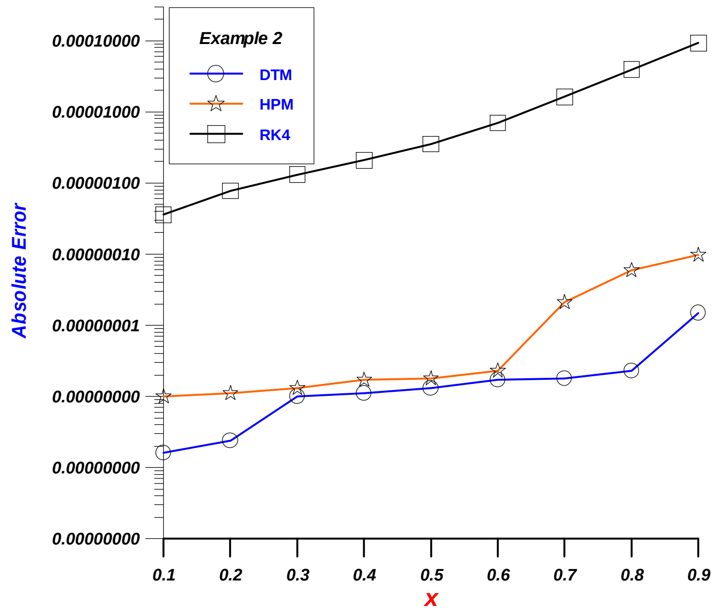

Figure (4) Errors for methods of Example (2)