Solving the Boundary Value Problems of Ordinary Differential Equation 4th order using RK4 and RK-Butcher Techniques

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Abstract
The two-point boundary value problems for the 4th order ordinary differential equations with a positive coefficient multiplying at least one of derivative terms are solved with two numerical methods. These numerical methods are the (Rung- Kutta of 4th Order) and (Rung–Kutta Butcher of 6th Order). The 4th order ordinary differential Equations problem had been transformed to pair of second Order differential equations, which were solved together by the suggested methods. An initial value of the dependent variable had been predicted and corrected to some error. The two studied methods were tested on a physical model problem from the literature for comparing results. Solutions were presented in Tables and figures. good agreements were appeared in applying the studied methods.

Keywords: Boundary Value Problems, Ordinary Differential Equation, RK4, RK-Butcher

حل مسائل القيمة الحدية ذات المعادلة التفاضلية من المرتبة الرابعة باستخدام

طريقتين 4th و 6th

الخلاصة
تم حل مسائل القيمة الحدية (BVPs) التي تحتوي على معادلات تفاضلية من المرتبة الرابعة والتي تكون على الأقل إحدى مشتقاتها معلمة بمعاملة ضرورية عند شروطها المحددة. إذ طُبقا كلاً من الطرق التالية (طريقة رانج – كوتا (Rung-Kutta) وطريقة من المرتبة السادسة (Rung - Kutta Butcher) لحل هذه المسائل في شروط معينة. تم تحويل المعادلة التفاضلية من المرتبة الرابعة إلى معادلات تفاضلية من المرتبة الثانية ليتم حلها معا باستخدام كل من الطرق كلاً على حده. يتم التعامل بالقيم الأولية للمتغيرات المعتمدة ويتم تغذيتها وتصحيحها وفق قيمة خطاً معين. تم تطبيق الطرقتين المذكورتين لحل بعض المسائل ثم قارنا النتائج الحاصلة مع طرق أخرى. دونت النتائج في جداول ومثلت بيانيا التي أظهرت نتائج جيدة.

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1. **Introduction**

Several models of mathematical physics and applied mathematics contain Boundary Value Problems BVPs in the 4th order ordinary differential equations (ODEs) [1]. Let consider the 4th order (ODEs) is of the following equation:

\[
\alpha \frac{d^4 y}{dx^4} + \beta \frac{d^2 y}{dx^2} + \gamma y = f(x, y)
\]  

(1)

\[y(0) = 0\]  

(2)

\[y'(0) = \xi\]  

(3)

\[y(1) = \psi\]  

(4)

Butcher [5] to solve BVPs with there conditions as solving Eqs.(1) under the conditions (2)-(5).

Special program is designed to apply the proposal methods. Some physical problems were studied before from [1],[2],[3], and [6] are solved by RK4 and RK-Butcher methods. Results are presented by tables and figures to compare the error with another solution which show good agreements.

2. **The Proposal Solution:**

To solve Eqs.(1) under the condition in (2)-(5) , it will be transform to two pair of 2nd (ODEs) by assuming:

\[
\frac{d^2 y}{dx^2} = M
\]  

(6)

Then Eqs.(1) becomes:

\[
\frac{d^2 M}{dx^2} + \frac{\beta}{\alpha} M = \frac{f(x, y) - \gamma y}{\alpha}
\]  

(7)

Where:

\[y(0) = 0\]  

(8)

\[M(0) = \xi\]  

(9)

\[y''(1) = \zeta\]  

(5)

where \(\alpha, \beta, \gamma\) and \(\xi, \psi, \zeta\) are constants and \(f(x, y)\) continuous and \(\frac{\partial f}{\partial y} \geq 0\) .

Different analytical and numerical methods are used to solve the 4th order (ODEs) this can be concerned by Kapur [1]. Some of the numerical methods applied by Ortner [2] which gave approximate solutions. Okey [3] used the GEM (Green Element Method) to solve these problems.

In this paper we proposed to apply Rung-Kutta (RK4)[4] and (RK-Butcher) techniques to solve BVPs with there conditions as solving Eqs.(1) under the conditions (2)-(5).

The proposal methods are:

2.1 ( Rung – Kutta of 4th Order ): The BVPs of 4th order (ODEs) is solved by Rung – Kutta of 4th Order (RK4) to solve Eqs.(6) and (7) after transforming them to two pair of 1st order (ODEs) as the following:

\[
\frac{dy}{dx} = p_1 = fnf_1(x, y, M, p_1, p_2)
\]  

(13)

\[
\frac{dp_1}{dx} = M = fnp_1(x, y, M, p_1, p_2)
\]  

(14)

\[
\frac{dM}{dx} = p_2 = fnf_2(x, y, M, p_1, p_2)
\]  

(15)
\[
\frac{dp_2}{dx} = fnp_2(x, y, M, p_1, p_2) = -\frac{\beta}{\alpha} M + (f(x, y) - \gamma y) / \alpha
\]  
(16)
The set of 1st order (ODEs) (13) to (16) are solved together from the following:
\[
y^{n+1} = y^n + \frac{1}{6}(k_{11} + 2k_{21} + 2k_{31} + k_{41})
\]  
(17)
\[
p_1^{n+1} = p_1^n + \frac{1}{6}(L_{11} + 2L_{21} + 2L_{31} + L_{41})
\]  
(18)
\[
M^{n+1} = M^n + \frac{1}{6}(k_{12} + 2k_{22} + 2k_{32} + k_{42})
\]  
(19)
\[
p_2^{n+1} = p_2^n + \frac{1}{6}(L_{22} + 2L_{32} + 2L_{42})
\]  
(20)
Constants
\[
k_{11}, k_{21}, k_{31}, k_{41}, L_{11}, L_{21}, L_{31}, L_{41},
\]
\[
k_{12}, k_{22}, k_{32}, k_{42}, L_{12}, L_{22}, L_{32}, L_{42},
\]
Eqs.(17) to (20) are calculated from the following:
\[
k_{11} = h: fnf_1(x, y, M, p_1, p_2)
\]  
(21)
\[
L_{11} = h: fnp_1(x, y, M, p_1, p_2)
\]  
(22)
\[
k_{12} = h: fnf_2(x, y, M, p_1, p_2)
\]  
(23)
\[
L_{12} = h: fnp_2(x, y, M, p_1, p_2)
\]  
(24)
\[
k_{21} = h: fnf_1(x + h/2, y + k_{11}/2, M + k_{12}/2, p_1 + L_{11}/2, p_2 + L_{12}/2)
\]  
(25)
\[
L_{21} = h: fnp_1(x + h/2, y + k_{11}/2, M + k_{12}/2, p_1 + L_{11}/2, p_2 + L_{12}/2)
\]  
(26)
\[
k_{22} = h: fnf_2(x + h/2, y + k_{11}/2, M + k_{12}/2, p_1 + L_{11}/2, p_2 + L_{12}/2)
\]  
(27)
\[
L_{22} = h: fnp_2(x + h/2, y + k_{11}/2, M + k_{12}/2, p_1 + L_{11}/2, p_2 + L_{12}/2)
\]  
(28)
\[
k_{31} = h: fnf_1(x + h/2, y + k_{21}/2, M + k_{22}/2, p_1 + L_{21}/2, p_2 + L_{22}/2)
\]  
(29)
\[
L_{31} = h: fnp_1(x + h/2, y + k_{21}/2, M + k_{22}/2, p_1 + L_{21}/2, p_2 + L_{22}/2)
\]  
(30)
\[
k_{32} = h: fnf_2(x + h/2, y + k_{21}/2, M + k_{22}/2, p_1 + L_{21}/2, p_2 + L_{22}/2)
\]  
(31)
\[
k_{41} = h: fnf_1(x + h, y + k_{31}, M + k_{32}, p_1 + L_{31}, p_2 + L_{32})
\]  
(32)
\[
L_{41} = h: fnp_1(x + h, y + k_{31}, M + k_{32}, p_1 + L_{31}, p_2 + L_{32})
\]  
(33)
\[
k_{42} = h: fnf_2(x + h, y + k_{31}, M + k_{32}, p_1 + L_{31}, p_2 + L_{32})
\]  
(34)
\[
L_{42} = h: fnp_2(x + h, y + k_{31}, M + k_{32}, p_1 + L_{31}, p_2 + L_{32})
\]  
(35)
\[
2.2 (Rung–Kutta Butcher of 6th Order):
The BVPs of 4th order (ODEs) is solved by Rung–Kutta Butcher of 6th Order (RK-Butcher) to solve Eqs.(6) and (7) after
transforming them to two pair of 1st order (ODEs) as the following:

\[ y^{n+1} = y^n + \frac{h}{90}(7k_{11} + 32k_{31} + 12k_{41} + 32k_{51} + 7k_{61}) \]  
(37)

\[ p_1^{n+1} = p_1^n + \frac{h}{90}(7L_{41} + 32L_{61} + 12L_{41} + 32L_{51} + 7L_{61}) \]  
(38)

\[ M^{n+1} = M^n + \frac{h}{90}(7k_{12} + 32k_{32} + 12k_{42} + 32k_{52} + 7k_{62}) \]  
(39)

\[ p_2^{n+1} = p_2^n + \frac{h}{90}(7L_{42} + 32L_{62} + 12L_{42} + 32L_{52} + 7L_{62}) \]  
(40)

Constants in Eqs.(37) to (40) are calculated

\[ k_{11} = fnf_1(x_n, y_n, M_n, p_{1n}, p_{2n}) \]  
(41)

\[ L_{41} = fnp_1(x_n, y_n, M_n, p_{1n}, p_{2n}) \]  
(42)

\[ k_{12} = fnf_2(x_n, y_n, M_n, p_{1n}, p_{2n}) \]  
(43)

\[ L_{42} = fnp_2(x_n, y_n, M_n, p_{1n}, p_{2n}) \]  
(44)

\[ k_{21} = fnf_1(x_n, y_n, M_n, p_{1n}, p_{2n}) + \frac{h}{4}L_{41} \]  
(45)

\[ L_{21} = fnp_1(x_n + \frac{h}{4}, y_n + \frac{hk_{11}}{4}, M_n + \frac{hk_{12}}{4}, p_{1n} + \frac{hL_{41}}{4}, p_{2n} + \frac{hL_{42}}{4}) \]  
(46)

\[ k_{22} = fnf_2(x_n, y_n, M_n, p_{1n}, p_{2n}) + \frac{h}{4}L_{42} \]  
(47)

\[ L_{22} = fnp_2(x_n + \frac{h}{4}, y_n + \frac{hk_{11}}{4}, M_n + \frac{hk_{12}}{4}, p_{1n} + \frac{hL_{41}}{4}, p_{2n} + \frac{hL_{42}}{4}) \]  
(48)
Solving the Boundary Value Problems of Ordinary Differential Equation 4th order Using RK4 and RK-Butcher Techniques

\[ k_{41} = \frac{f(x, y, M, p_{1n}, p_{2n})}{h} - \frac{h}{2}L_{21} + hL_{31} \]

\[ L_{41} = \frac{f(y, y' - \frac{h}{2}k_{21} + hk_{31}, M, p_{1n}, p_{2n})}{2} - \frac{h}{2}L_{22} + \frac{h}{8}L_{32} \]

\[ k_{42} = \frac{f(x, y, M, p_{1n}, p_{2n})}{h} - \frac{h}{2}L_{22} + \frac{h}{8}L_{32} \]

\[ L_{42} = \frac{f(y, y' - \frac{h}{2}k_{21} + hk_{31}, M, p_{1n}, p_{2n})}{2} - \frac{h}{2}L_{22} + \frac{h}{8}L_{32} \]

\[ k_{41} = \frac{f(x, y, M, p_{1n}, p_{2n})}{h} + \frac{3h}{16}L_{11} + \frac{9h}{16}L_{41} \]

\[ L_{51} = \frac{f(x, y, M, p_{1n}, p_{2n})}{h} + \frac{3h}{4}L_{11} + \frac{9h}{16}L_{41} \]

\[ k_{52} = \frac{f(x, y, M, p_{1n}, p_{2n})}{h} + \frac{3h}{16}L_{12} + \frac{9h}{16}L_{42} \]

\[ k_{51} = \frac{f(x, y, M, p_{1n}, p_{2n})}{h} + \frac{3h}{16}L_{12} + \frac{9h}{16}L_{42} \]

\[ k_{61} = \frac{f(x, y, M, p_{1n}, p_{2n})}{h} + \frac{3h}{16}L_{12} + \frac{9h}{16}L_{42} \]

\[ L_{61} = \frac{f(x, y, M, p_{1n}, p_{2n})}{h} + \frac{3h}{16}L_{12} + \frac{9h}{16}L_{42} \]

\[ (53) \]

\[ (54) \]

\[ (55) \]

\[ (56) \]

\[ (57) \]

\[ (58) \]

\[ (59) \]
Solving the Boundary Value Problems of Ordinary Differential Equation 4th order Using RK4 and RK-Butcher Techniques

\[ \frac{d^4 y}{dx^4} = 6e^{-4y} - \frac{12}{(1 + x)^4} \]  

(70)

where:
\[ y(0) = 0 \]  
(71)
\[ y'(0) = -1.0 \]  
(72)
\[ y(1) = \ln 2 \]  
(73)
\[ y'(1) = -0.25 \]  
(74)

This problem was studied by Okey [3] by applying the (Green Element Method). The Exact solution for this problem is \( y(x) = \ln(1 + x) \).

Results are presented in Table(1).

Example 1. To solve the nonlinear 4th order (ODEs) represented by:

\[ \frac{dy}{dx} = 2e^{2x}y_4^2 \]  
(75)

\[ \frac{dy_2}{dx} = y_1 - y_3 + \cos(x) - e^{2x} \]  
(76)

\[ \frac{dy_3}{dx} = y_2 - y_4 + e^{-x} - \sin(x) \]  
(77)

\[ \frac{dy_4}{dx} = -e^{-2x}y_1^2 \]  
(78)

\( y_1(0) = 1 \) , \( y_2(0) = 1 \) , \( y_3(0) = 0 \)
\( y_4(0) = 1 \)

3. Numerical applications:

Some of the physical problems are solved to assign the effectiveness and accuracy of the proposal methods. Results are presented in tables and figures.

Example 2. Consider the following set of 1st order nonlinear (ODEs) [6]:

\[ \frac{dy_1}{dx} = 2e^{x}y_4^2 \]  
(75)

\[ \frac{dy_2}{dx} = y_1 - y_3 + \cos(x) - e^{2x} \]  
(76)

\[ \frac{dy_3}{dx} = y_2 - y_4 + e^{-x} - \sin(x) \]  
(77)

\[ \frac{dy_4}{dx} = -e^{-2x}y_1^2 \]  
(78)

\( y_1(0) = 1 \) , \( y_2(0) = 1 \) , \( y_3(0) = 0 \)
\( y_4(0) = 1 \)
This problem was studied by Biazar [6] by applying the (Variational Iteration Method). The Exact Solution was given by:

\[ y_1(x) = e^{2x}, \quad y_2(x) = \sin x + \cos x, \]
\[ y_3(x) = \sin x, \quad y_4(x) = e^{-x}. \]

Results of calculating \( y_1(x), y_2(x), y_3(x), y_4(x) \) by using RK4 and RK-Butcher were presented in Tables (2) to (5). Figures (3), (4), (5), and (6) show errors between different methods.

4. Results & Discussion

The suggested methods RK4 and RK-Butcher for solving 4\textsuperscript{th} order (ODEs) gave good agreements comparing results of Example (1) with the Exact solution as shown from Table(1) and Figure(1).

Results of solving the problem in Example (2) by the three methods VIM, RK4, and RK-Butcher were presented in Tables (2) to (5) and Figures (3) to (6). RK4 was very accurate in the three methods for its least error. The VIM method was better than RK-Butcher due to its error. Finally the two proposal method RK4 and RK-Butcher gave good agreements in solving BVPs of 4\textsuperscript{th} order (ODEs) in the required conditions.

References

Table (1) Results of solving the problem in Example (1).

<table>
<thead>
<tr>
<th>X</th>
<th>Exact Solution</th>
<th>Rung-kutta Solution</th>
<th>Rung-kutta Error</th>
<th>RK-Butcher solution</th>
<th>RK-Butcher Error</th>
</tr>
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Table (2) Solutions of problem in Example(2) of  \( y_1(x) \):

<table>
<thead>
<tr>
<th>X</th>
<th>Exact Solution</th>
<th>Rung-Kutta Solution</th>
<th>Rung-Kutta Error</th>
<th>RK-Butcher solution</th>
<th>RK-Butcher Error</th>
<th>VIM solution</th>
<th>VIM Error</th>
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Table (3) Solutions of problem in Example(2) of  \( y_2(x) \):

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<th>Exact Solution</th>
<th>Rung-Kutta Solution</th>
<th>Rung-Kutta Error</th>
<th>RK-Butcher solution</th>
<th>RK-Butcher Error</th>
<th>VIM solution</th>
<th>VIM Error</th>
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Solving the Boundary Value Problems of Ordinary Differential Equation 4th order Using RK4 and RK-Butcher Techniques

**Table 4** Solutions of problem in Example(2) of $y_j(x)$:

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<th>Rung-Kutta Solution</th>
<th>Rung-Kutta Error</th>
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**Table 5** Solutions of problem in Example(2) of $y_j(x)$:

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<th>Rung-Kutta Solution</th>
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**Figure 1** Comparison of Error of Example(1).
Solving the Boundary Value Problems of Ordinary Differential Equation 4th order Using RK4 and RK-Butcher Techniques

Figure (2) Solving the problem in Example (1) by different methods

Figure (3) error between methods to calculate y1(x)

Figure (4) error between methods to calculate y2(x)
Solving the Boundary Value Problems of Ordinary Differential Equation 4th order Using RK4 and RK-Butcher Techniques

Figure (5) error between methods to calculate $y_3(x)$

Figure (6) error between methods to calculate $y_4(x)$