Approximate Solution of Fractional Integro-Differential Equations by Using Bernstein Polynomials

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ABSTRACT
In this paper, Bernstein piecewise polynomial is used to approximate the solution of the fractional integro-differential equations, in which the fractional derivative is described in the (Caputo) sense. Examples are considered to verify the effectiveness of the proposed derivation, and the approximate solutions guarantee the desired accuracy.

Keywords: - Fractional integro-differential equations, Bernstein polynomials.

INTRODUCTION
The concept of fractional or non-integer order derivation and integration can be traced back to the genesis of integer order calculus itself [1]. Almost every mathematical theory applicable to the study of non-integer order calculus was developed through the end of 19th century. However it is in the past hundred years that the most intriguing leaps in engineering and scientific application have been found. The calculation technique has to change in order to meet the requirement of physical reality in some cases. The use of fractional differentiation for the mathematical modeling of real world physical problems has been widespread in recent years, e.g. the modeling of earthquake, the fluid dynamic traffic model with fractional derivatives, measurement of viscoelastic material properties, etc, [2].

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There are several approaches to the generalization of the notion of differentiation of fractional orders, e.g. Riemann-Liouville, Grünwald–Letnikov, Caputo and generalized Functions approach [3].

Riemann-Liouville fractional derivative is the mostly used by mathematicians but this approach is not suitable for real world physical problems since it requires the definition of fractional order initial conditions, which have no physically meaningful explanation yet. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations [4].

Unlike the Riemann-Liouville approach which derives its definition from repeated integration, the Grünwald–Letnikov formulation approaches the problem from the derivative side; this approach is used in numerical algorithms.

A fractional Integro-Differential equation arises in modeling processes in applied sciences (Physics, Engineering, Finance, Biology …). Many problems in Viscoelasticity, Acoustics, Electromagnetics, Hydrology and other areas of application can be modeled by fractional Integro-Differential equations.


Mohammed [8] used the Homotopy Analysis method to solve the Fractional Integro-Differential equations.

In this paper we shall approximate the solution of the fractional order Integro-differential equation of the type:

\[ D^\alpha y(t) = f(t) + \int_a^b k(t,s)y(s) \, ds, \quad y(a) = y_a, \quad 0 < \alpha \leq 1 \]  

(1)

where \( D^\alpha \) is Caputo fractional derivative, \( \alpha \) is a parameter describing the order of the fractional derivative by using the Bernstein Polynomials. Bernstein Polynomials have been used recently to solve some linear as well as non-linear differential equations, ordinary and partial, approximately by Bhatta and Bhatti [9], and some Integral equations, by Mandal and Bhattacharya [10]. These polynomials defined over an interval forms a complete basis there, and each of them are positive and there sum is unity. Description of the properties of Bernstein Polynomials can be found in the paper of Bhatta and Bhatti [9]. Using these polynomials; we have obtained an approximate numerical solution of (1) which is acceptable when compared with the exact solution. It is important to notice that, all the computations needed in this paper are coded in Mathcad 14 software and the results are presented in tabulated form.
Many definitions of fractional calculus have been proposed [3]. Most frequently occurring are Caputo and Riemann-Liouville [4]. In this section, we shall discuss definitions and some basic properties of these two types of fractional derivatives.

**Abel-Riemann fractional integral and derivatives [7]**

The Abel-Riemann $A-R$ fractional integral of any order $\alpha > 0$ for a function $\psi(t)$ with $t \in \mathbb{R}^+$ is defined as:

$$J^\alpha \psi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} \psi(\tau) d\tau, t > 0, \alpha > 0$$  \hspace{1cm} (2)

The $A-R$ integrals possess the semigroup property:

$$J^\alpha J^\beta = J^{\alpha + \beta}, \text{for all } \alpha, \beta \geq 0$$  \hspace{1cm} (3)

The $A-R$ fractional derivative (of order $\alpha > 0$) is defined as the left inverse of the corresponding $A-R$ fractional integral, i.e.

$$D^\alpha J^\alpha = I$$  \hspace{1cm} (4)

For positive integer $m$ such that $m - 1 < \alpha \leq m$, $(D^m f^{m-\alpha}) f^\alpha = D^m (f^{m-\alpha} f^\alpha) = D^m f^m = I$, so that $D^\alpha = D^m f^{m-\alpha}$, i.e.

$$D^\alpha \psi(t) = \left\{ \begin{array}{ll}
\frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t \frac{\psi(\tau)}{(t-\tau)^{\alpha+1}} d\tau & \text{for } m - 1 < \alpha < m \\
\frac{d^m}{dt^m} \frac{\psi(t)}{\Gamma(\alpha)} & \text{for } \alpha = m
\end{array} \right.$$

(5)

Properties of the operators $J^\alpha$ and $D^\alpha$ can be found in [3], for $t > 0, \alpha \geq 0$, and $\gamma > -1$, we mention the following:

$$J^\alpha \tau^\gamma = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 1 + \alpha)} \tau^{\gamma + \alpha}, \quad D^\alpha \tau^\gamma = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 1 - \alpha)} \tau^{\gamma - \alpha}$$  \hspace{1cm} (6)

**Caputo fractional derivatives [7]**

In the late sixtieth decade an alternative definition of fractional derivative was introduced by Caputo. Caputo and Mirandi used this definition in their work on the theory of viscoelasticity. According to Caputo’s definition $D^\alpha \psi(t) = f^{m-\alpha} D^m$ for $m - 1 < \alpha \leq m$, i.e.

$$D^\alpha \psi(t) = \left\{ \begin{array}{ll}
\frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{\psi(\tau)}{(t-\tau)^{\alpha+1}} d\tau & \text{for } m - 1 < \alpha < m \\
\frac{d^m}{dt^m} \frac{\psi(t)}{\Gamma(\alpha)} & \text{for } \alpha = m
\end{array} \right.$$

(6)

One of the basic properties of the Caputo fractional derivative is:

$$J^\alpha D^\alpha \psi(t) = \psi(t) - \sum_{k=0}^{m-1} \psi^{(k)}(0^+) \frac{t^k}{k!}$$  \hspace{1cm} (7)
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Caputo’s fractional differentiation is a linear operation, similar to integer order differentiation, $D^\alpha_t[\lambda f(t) + \mu g(t)] = \lambda D^\alpha_t f(t) + \mu D^\alpha_t g(t)$, where $\lambda$ and $\mu$ are constants.

**BERNSTEIN POLYNOMIALS**

The general form of the Bernstein polynomials of $n^{th}$ degree over the interval is defined by

$$B_{i,n}(t) = \binom{n}{i} \frac{(t-a)^i (b-t)^{n-i}}{(b-a)^n}, a \leq t \leq b, i = 0,1,2,\ldots, n$$

Note that for more properties one can see [12].

**FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION APPROXIMATE SOLUTION FORMULATION**

In this section, we shall derive the approximate solution of the fractional Integro-differential equation by using the Bernstein polynomials as follows:

Consider the Fractional Integro-Differential equation of the form,

$$D^\alpha_t y(x) = f(x) + \int_a^b k(x,t)y(t)dt , 0 < \alpha \leq 1$$

where $D^\alpha_t$ is the Caputo fractional derivative of order $\alpha$ and $f(x), k(x,t)$ are given functions.

Our approach begins by taking the fractional integration to both sides of equation (8) we get:

$$y(x) = y(0) + I^\alpha f(x) + I^\alpha \left( \int_a^b k(x,t)y(t)dt \right)$$

To determine the approximate solution of (8), we use the Bernstein polynomial basis on $[a,b]$ as

$$y(x) = \sum_{i=0}^n a_i B_{i,n}(x)$$

where $a_i (i = 0,1,\ldots,n)$ are unknown constants to be determined.

Substituting equation (10) into equation (9), we obtain:

$$\sum_{i=0}^n a_i B_{i,n}(x) = y(0) + I^\alpha f(x) + I^\alpha \left( \int_a^b k(x,t) \sum_{i=0}^n a_i B_{i,n}(t) dt \right)$$

Hence

$$\sum_{i=0}^n a_i B_{i,n}(x) = y(0) + I^\alpha f(x)$$

where $\phi(x) = \int_a^b k(x,t) B_{i,n}(t) dt$
thus, we have
\[ \sum_{i=0}^{n} a_i \left[ B_{i,n}(x) - I^\alpha \phi(x) \right] = y(0) + I^\alpha f(x) \]  
(11)
Now, we put \( x = x_j, j = 0, 1, \cdots, n \) into equation (11), \( x_j^\alpha \) are being chosen as suitable distinct points in \((a, b)\), putting \( x = x_j \) we obtain the linear system:
\[ \sum_{i=0}^{n} a_i \alpha_{ij} = f_j \quad j = 0, 1, \cdots, n \]  
(12)
where \( \alpha_{ij} = B_{i,n}(x_j) - I^\alpha \phi(x_j) \) and \( f_j = y(0) + I^\alpha f(x_j) \).
The linear system (12) can be easily solved by standard methods for the unknown constants \( a_i^\alpha \). These \( a_i \) (\( i = 0, 1, 2, \cdots, n \)) are then used in equation (10) to obtain the unknown function \( y(x) \) approximately.

**ILLUSTRATIVE EXAMPLES**
In this section, we shall give some illustrative examples in order to clarify our approach.

**Example(1)**
Consider the fractional Integro-Differential Equation:
\[ D^\alpha y(x) = 1 - \frac{1}{3} x + \int_0^1 xty(t)dt, y(0) = 0, 0 < \alpha \leq 1 \]  
(13)
By taking the fractional integration for both sides of the above equation, we get:
\[ y(x) = y(0) + I^\alpha \left( 1 - \frac{1}{3} x \right) + I^\alpha \left( \int_0^1 xty(t)dt \right) \]  
(14)
To determine the approximate solution of eq.(13), we set \( y(x) = \sum_{i=0}^{3} a_i B_{i,3}(x) \) and after substituting it into eq.(14), we get:
\[ \sum_{i=0}^{3} a_i B_{i,3}(x) = y(0) + I^\alpha \left( 1 - \frac{1}{3} x \right) + I^\alpha \left( \int_0^1 xt \sum_{i=0}^{3} a_i B_{i,3}(t) dt \right) \]
So,
\[ a_0(1 - x)^3 + a_13x(1 - x)^2 + a_23x^2(1 - x) + a_3x^3 \]
\[ - I^\alpha \left[ a_0x \int_0^1 t(1 - t)^3 dt + a_1x \int_0^1 3t^2(1 - t)^2 dt \right. \]
\[ \left. + a_2x \int_0^1 3t^3(1 - t) dt + a_3x \int_0^1 t^4 dt \right] = I^\alpha \left( 1 - \frac{1}{3} x \right) \]
Hence,
\[ a_0 \left[ \frac{(1 - x)^3 - 0.05 \frac{\Gamma(2)}{\Gamma(\alpha+2)} x^{\alpha+1}}{\Gamma(\alpha+1)} \right] + a_1 \left[ 3x(1 - x)^2 - 0.1 \frac{\Gamma(2)}{\Gamma(\alpha+2)} x^{\alpha+1} \right] + \]
\[ a_2 \left[ 3x^2(1 - x) - 0.15 \frac{\Gamma(2)}{\Gamma(\alpha+2)} x^{\alpha+1} \right] + a_3 \left[ x^3 - 0.2 \frac{\Gamma(2)}{\Gamma(\alpha+2)} x^{\alpha+1} \right] = \]
(15)
Case (1)
when \( \alpha = 1 \) then equation (15), will be:
\[
a_0(1 - x)^3 + 3a_1 x(1 - x)^2 + 3a_2 x^2(1 - x) + a_3 x^3 - 0.05a_0 \int_0^x t \, dt - 0.1a_1 \int_0^x t \, dt - 0.15a_2 \int_0^x t \, dt - 0.2a_3 \int_0^x t \, dt = \int_0^x t \, dt - \frac{1}{3} \int_0^x t \, dt.
\]
or
\[
a_0 \left[(1 - x)^3 - \frac{0.05}{2} x^2 \right] + a_1 \left[3x(1 - x)^2 - \frac{0.1}{2} x^2 \right] + a_2 \left[3x^2(1 - x) - \frac{0.15}{2} x^2 \right] + a_3 \left[x^3 - \frac{0.2}{2} x^2 \right] = x - \frac{x^2}{2}
\]
which may be written as:
\[
a_0 \left[(1 - x)^3 - \frac{0.05}{2} \Gamma(2) \Gamma(2.5)^{1.5} \right] + a_1 \left[3x(1 - x)^2 - 0.1 \Gamma(2) \Gamma(2.5)^{1.5} \right] + a_2 \left[3x^2(1 - x) - 0.15 \Gamma(2) \Gamma(2.5)^{1.5} \right] + a_3 \left[x^3 - 0.2 \Gamma(2) \Gamma(2.5)^{1.5} \right] = \\frac{1}{\Gamma(1.5)} x^{0.5} - \frac{1}{\Gamma(1.5)} \Gamma(2) \Gamma(2.5)^{1.5}
\]
Following table (1) represent a comparison between the approximate solution of example (1) when \( \alpha = 1 \) with the exact solution \( y = x \).

<table>
<thead>
<tr>
<th>x</th>
<th>Approximate Solution</th>
<th>Exact Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.09996</td>
<td>0.1</td>
</tr>
<tr>
<td>0.2</td>
<td>0.1999</td>
<td>0.2</td>
</tr>
<tr>
<td>0.3</td>
<td>0.2999</td>
<td>0.3</td>
</tr>
<tr>
<td>0.4</td>
<td>0.3996</td>
<td>0.4</td>
</tr>
<tr>
<td>0.5</td>
<td>0.4966</td>
<td>0.5</td>
</tr>
<tr>
<td>0.6</td>
<td>0.5991</td>
<td>0.6</td>
</tr>
<tr>
<td>0.7</td>
<td>0.6982</td>
<td>0.7</td>
</tr>
<tr>
<td>0.8</td>
<td>0.7968</td>
<td>0.8</td>
</tr>
<tr>
<td>0.9</td>
<td>0.8948</td>
<td>0.9</td>
</tr>
</tbody>
</table>

Table (1) comparison between the approximate solution of example (1) when \( \alpha = 1 \) with the exact solution.

Case (2)
when \( \alpha = 0.5 \) into equation (15), we have:
\[
a_0(1 - x)^3 + 3a_1 x(1 - x)^2 + 3a_2 x^2(1 - x) + a_3 x^3 - \int_0^{0.5}(0.05a_0 x) - \int_0^{0.5}(0.1a_1 x) - \int_0^{0.5}(0.15a_2 x) - \int_0^{0.5}(0.2a_3 x)
\]
which may be written as:
\[
a_0 \left[(1 - x)^3 - 0.05 \Gamma(2) \Gamma(2.5)^{1.5} \right] + a_1 \left[3x(1 - x)^2 - \frac{0.1}{2} \Gamma(2) \Gamma(2.5)^{1.5} \right] + a_2 \left[3x^2(1 - x) - \frac{0.15}{2} \Gamma(2) \Gamma(2.5)^{1.5} \right] + a_3 \left[x^3 - \frac{0.2}{2} \Gamma(2) \Gamma(2.5)^{1.5} \right] = \\frac{1}{\Gamma(1.5)} x^{0.5} - \frac{1}{\Gamma(1.5)} \Gamma(2) \Gamma(2.5)^{1.5}
\]
now, substituting \( x = 0.1, 0.2, 0.3, \) and \( 0.4 \) into eq.(17), respectively, we will a linear system, that has the following solution,
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\[ a_0 = 0.158, a_1 = 0.937, a_2 = 0.691, \text{ and } a_3 = 1.987 \]

Thus, the approximate solution of eq.(13), when \( \alpha = 0.5 \) becomes:

\[ y(x) = 0.158(1 - x)^3 + 0.937 \times 3x(1 - x)^2 + 0.691 \times 3x^2(1 - x) + 1.987x^3 \]

Following table (2) represent the approximate solution of example (1) when \( \alpha = 0.5 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>Approximate Solution</th>
<th>Residual</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.364</td>
<td>0.04</td>
</tr>
<tr>
<td>0.2</td>
<td>0.523</td>
<td>0.01</td>
</tr>
<tr>
<td>0.3</td>
<td>0.652</td>
<td>0.01</td>
</tr>
<tr>
<td>0.4</td>
<td>0.765</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.879</td>
<td>0.04</td>
</tr>
<tr>
<td>0.6</td>
<td>1.008</td>
<td>0.14</td>
</tr>
<tr>
<td>0.7</td>
<td>1.168</td>
<td>0.31</td>
</tr>
<tr>
<td>0.8</td>
<td>1.374</td>
<td>0.59</td>
</tr>
<tr>
<td>0.9</td>
<td>1.642</td>
<td>0.96</td>
</tr>
</tbody>
</table>

Table (2) the approximate solution of example (1) when \( \alpha = 0.5 \).

**Case (3)**

when \( \alpha = 0.25 \) into eq.(15), we have:

\[ a_0(1 - x)^3 + 3a_1x(1 - x)^2 + 3a_2x^2(1 - x) + a_3x^3 - t^{0.25}(0.05a_0x) - t^{0.25}(0.1a_1x) - t^{0.25}(0.15a_2x) - t^{0.25}(0.2a_3x) = t^{0.25}(1) - t^{0.25}(1/3x). \]

which may be written as:

\[
\begin{align*}
& a_0 \left[ (1 - x)^3 - 0.05 \frac{\Gamma(2)}{\Gamma(2.25)} x^{1.25} \right] + a_1 \left[ 3x(1 - x)^2 - 0.1 \frac{\Gamma(2)}{\Gamma(2.25)} x^{1.25} \right] + \\
& a_2 \left[ 3x^2(1 - x) - 0.15 \frac{\Gamma(2)}{\Gamma(2.25)} x^{1.25} \right] + a_3 \left[ x^3 - 0.2 \frac{\Gamma(2)}{\Gamma(2.25)} x^{1.25} \right] = \\
& \frac{1}{\Gamma(1.25)} x^{0.25} - \frac{1}{3 \Gamma(2.25)} x^{1.25}.
\end{align*}
\]

(18)

now, substituting \( x = 0.1, 0.2, 0.3, \text{ and } 0.4 \) into eq.(18), respectively, we will get a linear system, that has the following solution,

\[ a_0 = 0.441, a_1 = 1.218, a_2 = 0.751, \text{ and } a_3 = 2.345 \]

Thus, the approximate solution of equation (13), when \( \alpha = 0.25 \) becomes:

\[ y(x) = 0.441 \times (1 - x)^3 + 1.218 \times 3x(1 - x)^2 + 0.751 \times 3x^2(1 - x) + 2.345 \times x^3 \]
Following table (3) represent the approximate solution of example (1) when $\alpha = 0.25$,

<table>
<thead>
<tr>
<th>$x$</th>
<th>Approximate Solution</th>
<th>Residual</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.6401</td>
<td>0.02</td>
</tr>
<tr>
<td>0.2</td>
<td>0.7844</td>
<td>0.01</td>
</tr>
<tr>
<td>0.3</td>
<td>0.8937</td>
<td>0</td>
</tr>
<tr>
<td>0.4</td>
<td>0.9878</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>1.0866</td>
<td>0.03</td>
</tr>
<tr>
<td>0.6</td>
<td>1.2100</td>
<td>0.1</td>
</tr>
<tr>
<td>0.7</td>
<td>1.3776</td>
<td>0.26</td>
</tr>
<tr>
<td>0.8</td>
<td>1.6095</td>
<td>0.5</td>
</tr>
<tr>
<td>0.9</td>
<td>1.9253</td>
<td>0.86</td>
</tr>
</tbody>
</table>

Table (3) the approximate solution of example (1) when $\alpha = 0.25$.

Following figure (1) represent the approximate solution of example (1) for different values $\alpha = 1, 0.5, 0.25$.

Figure (1) the approximate solution of example (1) for $\alpha = 1, 0.5, 0.25$.

**Example (2)**

Consider the fractional Integro-Differential Equation:

$$D^\alpha y(x) = xe^x + e^x - x + \int_0^1 x y(t)dt, y(0) = 0, 0 < \alpha \leq 1 \quad (19)$$

By taking the fractional Integration to both sides of eq.(19), we have:
\[ y(x) = I^\alpha(xe^x + e^x - x) + I^\alpha \left( \int_0^1 x y(t) \, dt \right) \]  

(20)

To approximate the solution of eq.(19), we set \( y(x) = \sum_{i=0}^{3} a_i B_i(x) \), and substituting into eq.(20), we get:

\[ \sum_{i=0}^{3} a_i B_i(x) - I^\alpha \left( \int_0^1 x \sum_{i=0}^{3} a_i B_i(t) \, dt \right) = I^\alpha(xe^x + e^x - x), \]

or

\[
a_0(1 - x)^3 + 3a_1 x(1 - x)^2 + 3a_2 x^2(1 - x) + a_3 x^3 - I^\alpha \left[ a_0 x \int_0^1 (1 - t)^3 \, dt + 3a_1 x \int_0^1 t(1 - t)^2 \, dt + 3a_2 x \int_0^1 t^2(1 - t) \, dt + a_3 x \int_0^1 t^3 \, dt \right] = I^\alpha(xe^x + e^x - x),
\]

and in order to avoid the difficulty of evaluating the fractional integration of \( e^x \), we shall substitute its maclurian series up five terms to get:

\[
a_0 \left[ (1 - x)^3 - \frac{0.25}{\Gamma(\alpha+2)} x^{\alpha+1} \right] + a_1 \left[ 3x(1 - x)^2 - \frac{0.25}{\Gamma(\alpha+2)} x^{\alpha+1} \right] + a_2 \left[ 3x^2(1 - x) - \frac{0.25}{\Gamma(\alpha+2)} x^{\alpha+1} \right] + a_3 \left[ x^3 - \frac{0.25}{\Gamma(\alpha+2)} x^{\alpha+1} \right] = I^\alpha \left( 1 + x + \frac{3}{2} x^2 + \frac{4}{6} x^3 + \frac{1}{6} x^4 \right)
\]

(21)

Case (1)

When \( \alpha = 1 \) into eq.(21) and after substituting \( x = 0.1, 0.2, 0.3, \) and \( 0.4 \) respectively, we get a linear system that has the following solution,

\[
a_0 = -2.377 \times 10^{-3}, a_1 = 0.348, a_2 = 0.933, \text{ and } a_3 = 2.819.
\]

Thus, the approximate solution of eq.(19), when \( \alpha = 1 \) becomes:

\[
y(x) = -2.377 \times 10^{-3} \times (1 - x)^3 + 0.348 \times 3x(1 - x)^2 + 0.933 \times 3x^2(1 - x) + 2.819 \times x^3
\]

Following table (4) represent a comparison between the approximate solution of example (2) with the exact solution \( y = xe^x \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>Approximate Solution</th>
<th>Exact Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.111</td>
<td>0.11052</td>
</tr>
<tr>
<td>0.2</td>
<td>0.245</td>
<td>0.24428</td>
</tr>
<tr>
<td>0.3</td>
<td>0.405</td>
<td>0.40496</td>
</tr>
<tr>
<td>0.4</td>
<td>0.599</td>
<td>0.59673</td>
</tr>
<tr>
<td>0.5</td>
<td>0.832</td>
<td>0.82436</td>
</tr>
<tr>
<td>0.6</td>
<td>1.112</td>
<td>1.09327</td>
</tr>
<tr>
<td>0.7</td>
<td>1.444</td>
<td>1.40963</td>
</tr>
<tr>
<td>0.8</td>
<td>1.835</td>
<td>1.78043</td>
</tr>
<tr>
<td>0.9</td>
<td>2.291</td>
<td>2.21364</td>
</tr>
</tbody>
</table>

Table (4) comparison between the approximate solution of example (2) when \( \alpha = 1 \) with the exact solution.
Case (2) when \( \alpha = 0.5 \) into eq.(21) and after substituting \( x = 0.1, 0.2, 0.3, \) and 0.4 respectively, we shall get a linear system of equations which has the solutions:

\[
\alpha_0 = 0.145, \alpha_1 = 1.115, \alpha_2 = 1.966, \text{ and } \alpha_3 = 5.362
\]

hence the approximate solution of equation (19), when \( \alpha = 0.5 \) becomes:

\[
y(x) = 0.145(1 - x)^3 + 1.115 \times 3x(1 - x)^2 + 1.966 \times 3x^2(1 - x) + 5.362x^3
\]

Following table (5) represent the approximate solution of example (2) when \( \alpha = 0.5 \):

<table>
<thead>
<tr>
<th>x</th>
<th>Approximate Solution</th>
<th>Residual</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.435</td>
<td>0.04</td>
</tr>
<tr>
<td>0.2</td>
<td>0.734</td>
<td>0.01</td>
</tr>
<tr>
<td>0.3</td>
<td>1.058</td>
<td>0</td>
</tr>
<tr>
<td>0.4</td>
<td>1.422</td>
<td>0.01</td>
</tr>
<tr>
<td>0.5</td>
<td>1.844</td>
<td>0.05</td>
</tr>
<tr>
<td>0.6</td>
<td>2.338</td>
<td>0.14</td>
</tr>
<tr>
<td>0.7</td>
<td>2.921</td>
<td>0.28</td>
</tr>
<tr>
<td>0.8</td>
<td>3.608</td>
<td>0.47</td>
</tr>
<tr>
<td>0.9</td>
<td>4.417</td>
<td>0.73</td>
</tr>
</tbody>
</table>

Table (5) approximate solution of example (2) when \( \alpha = 0.5 \).

Case (3) when \( \alpha = 0.25 \) into eq.(21) and after substituting \( x = 0.1, 0.2, 0.3, \) and 0.4 respectively, we shall get a linear system of equations which has the solutions:

\[
\alpha_0 = 0.417, \alpha_1 = 1.791, \alpha_2 = 3.073, \text{ and } \alpha_3 = 6.992
\]

hence the approximate solution of eq.(19), when \( \alpha = 0.5 \) becomes:

\[
y(x) = 0.417(1 - x)^3 + 1.791 \times 3x(1 - x)^2 + 3.073 \times 3x^2(1 - x) + 6.992x^3
\]

Following table (6) represent the approximate solution of example (2) when \( \alpha = 0.25 \),

<table>
<thead>
<tr>
<th>x</th>
<th>Approximate Solution</th>
<th>Residual</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.8292</td>
<td>0.02</td>
</tr>
<tr>
<td>0.2</td>
<td>1.2522</td>
<td>0.01</td>
</tr>
<tr>
<td>0.3</td>
<td>1.7024</td>
<td>0</td>
</tr>
<tr>
<td>0.4</td>
<td>2.1963</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>2.7501</td>
<td>0.01</td>
</tr>
<tr>
<td>0.6</td>
<td>3.3803</td>
<td>0.06</td>
</tr>
<tr>
<td>0.7</td>
<td>4.1032</td>
<td>0.14</td>
</tr>
<tr>
<td>0.8</td>
<td>4.9352</td>
<td>0.27</td>
</tr>
<tr>
<td>0.9</td>
<td>5.8927</td>
<td>0.44</td>
</tr>
</tbody>
</table>
Table (6) the approximate solution of example (2) when $\alpha = 0.25$.
The approximate solution of example (2) will be represented by figure (2) for values of $\alpha = 1, 0.5,$ and $0.25$.

![Figure (2) The Approximate Solution of Example (2) for $\alpha=1,0.5,$ and $0.25$.](attachment:figure2.png)

**CONCLUSIONS**

In this paper a very simple and straight forward method, based on approximation of the unknown function of the fractional integro-differential equation on the Bernstein polynomial basis is presented, its use, produces acceptable results if it is compared with the exact solution when $\alpha = 1$, also it is acceptable for $\alpha = 0.5,$ and $0.25$ since the Residual is less than one for all values of $x$.

**REFERENCES**

Approximate Solution of Fractional Integro-Differential Equations by Using Bernstein Polynomials