Completeness of the Cartesian Product of Two Complete Fuzzy Normed Spaces

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ABSTRACT

In this paper it was proved that the Cartesian product of two fuzzy normed spaces is a fuzzy normed space then it was proved that the Cartesian product of two complete fuzzy normed spaces is again a complete fuzzy normed space.

Key Words: Fuzzy normed space, Cartesian product, Cauchy sequence, Complete fuzzy normed space.

الكمال لحاصل الضرب الديكارتي لفضائيين قياسيين ضبابيين كاملين

الخلاصة

في هذا البحث برهنا ان الضرب الديكارتي لفضائيين قياسيين ضبابيين هو فضاء قياسي ضبابي ثم بعد ذلك برهناحاصل الضرب الديكارتي لفضائيين قياسيين ضبابيين كاملين هو فضاء قياسي ضبابي كامل .

INTRODUCTION

The theory of fuzzy sets was introduced by Zadeh in 1965 [1] . The idea of fuzzy norm was initiated by Katsaras in [2] . Felbin in [3] defined a fuzzy norm on a linear space whose associated fuzzy metric is of Kaleva-Seikkala type [4]. Cheng and Mordeson in [5] introduced an idea of a fuzzy norm on a linear space whose associated metric is Kramosil and Michalek type in [6]. Bag and Samanta in [7] gave a definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [6]. They also studied some properties of fuzzy norm in [7] and [8] .Bag and Samanta discussed the notions of convergent sequence and Cauchy sequence in fuzzy normed linear space in [8]. The study of products spaces in probabilistic framework was initiated by Istratescu and Vaduva [9] and subsequently by Egbert in [10] , Alsina in [11] and Alsina and Schweizer in [12] . The main purpose of this paper is to introduce the concepts of the Cartesian product of fuzzy normed space then we will prove that the Cartesian product of two complete fuzzy normed spaces is also a complete fuzzy normed space .

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S1: Preliminaries

In this section the fundamental concepts related to this paper are given for completeness of the work.

Definition 1.1: [13]

A binary operation $*$:[0,1] × [0,1] → [0,1] is a continuous triangular norm if for all a,b,c,e ∈ [0,1] the following conditions hold :

1- $a * b = b * a$

2- $a * 1 = a$

3- $(a * b) * c = a * (b * c)$

4- $a * b \leq c * e$ whenever $a \leq c$ and $b \leq e$

In the sequel we shall refer to the triangular norm as t-norm .

Example 1.2 :[13]

Define $a * b = ab$, for all a,b∈ [0,1] where ab is the usual multiplication in [0,1] for all a,b ∈ [0,1] . It follows that $*$ is a continuous t-norm .

Example 1.3 :[13]

Define $a * b = \min \{a, b\}$ for all a, b ∈ [0,1] . Then $*$ is a continuous t norm .

Definition 1.4: [13]

Given an arbitrary set X, a fuzzy set M is a function from X to the unit interval [0,1] .

Definition 1.5: [8]

The triple (X,N,$*$) is said to be a fuzzy normed space where X is a linear space over the Field $\mathbb{K}$ (field of real or complex numbers) and $*$ is a continuous t-norm and N is a fuzzy set on X× [0,∞) satisfying the following conditions :

(FN1) for all x∈ X , N(x,0) = 0 .

(FN2) for all t> 0, N(x,t) = 1 if and only if x = 0 .

(FN3) for each α≠ 0 ∈ $\mathbb{K}$ and t> 0, $N(\alpha x,t) = N(x,\frac{t}{|\alpha|})$.

(FN4) for all x,y ∈ X and for all s> 0 , t> 0 , $N(x+y,s+t) \geq N(x,s) * N(y,t)$.

(FN5) for all x ∈ X , N(x,•) : [0,∞) → [0,1] is continuous function of t .

Remark 1.6:[8]

(i) Condition FN5 in Definition 1.5 means that for each x∈ X there is a function N_x: [0,∞)→ [0,1] , t→ N(x,t) .

(ii) Note that N(x,t) may be considered as the degree of nearness between x and 0 with respect to t .

Example 1.7:[8]

Let (X,||.||) be an ordinary norm , define $N(x,t) = \frac{t}{t+||x||}$ and $a * b = ab$ for all a, b∈ [0,1] . Then (X,N,$*$) is a fuzzy normed space which is called the standard fuzzy normed space.

Remark 1.8:[8]

Note that Example 1.7 holds even with the continuous t-norm $a * b = \min \{a , b \}$.

Example 1.9:[8]

Let X = $\mathbb{R}$. Define $a * b =ab$ for all a, b ∈ [0, 1] and $N(x, t) = [\exp\left(\frac{|x|}{t}\right)]^{-1}$ for all x ∈ X, t> 0.

Then (X, N,$*$) is a fuzzy normed space.

Remark 1.10:[8]

Note that in example 1.9 one may replace ($\mathbb{R}$,|.|) by any ordinary normed space , also this example is a fuzzy normed space with t-norm $a * b = \min \{ a , b \}$ for all a , b ∈ [0,1] .

Lemma 1.11:[9]

N(x,•) is a nondecreasing function of t , for all x ∈X .

Definition 1.12:[9]

Let X and Y be any two sets ,the Cartesian product is denoted by X×Y and is defined by X×Y = { (x,y) : x ∈ X , y ∈Y } .

Definition 1.13: [8]

Let (X, N,∗) be a fuzzy normed space. A sequence { x_n } in X is said to be convergent if there exists x in X such that $\lim_{n\to\infty} N(x_n - x, t) = 1$,for each t > 0 .

Definition 1.14:[8]

Let (X, N,∗) be a fuzzy normed space. A sequence { x_n } is said to be Cauchy sequence if $\lim_{n\to\infty} N(x_{n+p} - x_n, t) = 1$, for each t > 0 and p = 1,2 ,3,….. .

Definition 1.15:[8]

A fuzzy normed space (X, N,∗) is said to be complete if every Cauchy sequence is convergent.

Definition 1.16:[9]

Let o and ∗ are continuous t-norm then o is said to be stronger than ∗ if for each a,b,c,d ∈ [0,1] . Then (a∗ b) o (c ∗d) ≥ (a o c) ∗ (b o d).

Lemma 1.17: [10]

If o is stronger than ∗ then o ≥ ∗ that is a o b ≥ c ∗ d, for all a, b, c, d ∈ [0,1]

S2:Completeness of X× Y

We now define Cartesian product of two fuzzy normed spaces then we prove that theCartesian product of two fuzzy normed spaces is again a fuzzy normed space .Finally we prove that the Cartesian product of two complete fuzzy normed spaces is again a complete fuzzy normed space.

Definition 2.1

Let (X,N_1,∗) and (Y,N_2,∗) be two fuzzy normed spaces defined with same continuous t-norm ∗ .Let o be a continuous t-norm .The Cartesian product of (X,N_1,∗) and (Y,N_2, ∗) is the product space (X× Y,N,*) where X× Y is the Cartesian product of the sets X and Yand N a is mapping from (X× [0,∞))× (Y× [0,∞)) into [0,1] given by : $N((x_1,y_1),t+s) = N_1(x_1,t)$ o $N_2(y_1,s)$, for all (x_1,y_1) ∈ X× Y , t> 0, s> 0 .

Theorem 2.2

Let (X, N1,∗) and (Y, N2,∗) be two fuzzy normed spaces under the same continuous t-norm .Then (X×Y, N,∗) is a fuzzy normed space.

Proof

For each (x_1, y1),(x,y) in X×Y ,we have

(FN_1) $N((x_1,y_1), 0) = N_1(x_1,0) * N_2(y_1,0) = 0$, since $N_1(x_1,0) = 0$ and $N_2(y_1,0) = 0$

(FN_2)$N_1(x,s) = 1$ for each s > 0 if and only if x = 0 ,also $N_2(y,t) = 1$, for each t > 0 if and only if y = 0 .Together $N_1(x,s) * N_2(y,t) = 1$, for each t > 0 ,s> 0 if and only if (x,y) = (0,0) . Hence N((x,y),,s+t) = 1 for each t > 0 ,s> 0 if and only if (x,y) = (0,0).

(FN_3)If c ≠ 0 ∈ 𝕂 then for each t > 0 , s > 0 , $N_1(cx,s) = N_1(x, \frac{s}{|c|})$ and $N_2(cy,t) = N_2(y, \frac{t}{|c|})$. Now for each t> 0, s> 0 N(c(x,y),s+t) = N(cx,cy,s+t) = $N_1(cx,s) * N_2(cy,t) = N_1(x, \frac{s}{|c|}) * N_2(y, \frac{t}{|c|})$ $N((x,y), \frac{s+t}{|c|})$.

(FN_4) $N_1(x+x_1,s+t) \geq N_1(x,,s) * N_!(x_1,t)$ for each s> 0,t> 0 .Also $N_2(y+y_1,s+t) \geq N_2(y,s) * N_2(y_1,t)$ for each s > 0,t > 0 .Now for each s> 0,t > 0N $((x,y)+(x_1,y_1),2s+2t)$

$= N_1(x+x_1,2s) \ast N_2(y+y_1,2t) \geq [N_1(x,s) \ast N_1(x_1,s)] \ast [N_2(y,t) \ast N_2(y_1,t)] \geq [N_1(x,s) \ast N_2(y,t)] \ast [N_1(x_1,s) \ast N_2(y_1,t)] \geq N((x,y),s+t) \ast N((x_1,y_1),s+t)$.

(FN5) for all $(x,y) \in X\times Y$ we have $N_1(x,\bullet): [0,\infty) \to [0,1]$ is continuous and $N_2(y,\bullet) : [0,\infty) \to [0,1]$ is continuous .Now let $\{t_n\}$ be a sequence in $[0,\infty)$ converge to t in $[0,\infty)$ and $\{s_n\}$ converge to s in $[0,\infty)$ that $\lim_{n\to\infty} t_n = t$ and $\lim_{n\to\infty} s_n = s$.

Hence $N((x,y),t_n+s_n) = N_1(x,t_n) \ast N_2(y,s_n) \to N_1(x,t) \ast N_2(y,s) = N((x,y),t+s)$ since $N_1(x,t_n) \to N_1(x,t)$ and $N_2(y,s_n) \to N_2(y,s)$.

Theorem 2.3

Let (X, N1,$\ast$) and (Y, N2,$\ast$) be two fuzzy normed spaces under the same t-norm $\ast$.If the t-norm o is stronger than $\ast$ then the product (X× Y,N,o) is a fuzzy normed space .

Proof

It is suffices to prove (FN4) and (FN5):

Let (x_1,y_1) , (x_2,y_2) X× Y .Then $N((x_1,y_1) + (x_2,y_2), 2(t+s)) = N((x_1+x_2,y_1+y_2),2(t+s))$
$= N_1(x_1+x_2,2t) \text{ o } N_2(y_1+y_2,2s) \geq [N_1(x_1,t) \ast N_1(x_2,t)] \text{o} [N_2(y_1,s) \ast N_2(y_2,s)] \geq [N_1(x_1,t) \text{ o } N_2(y_1,s)] \ast [N_1(x_2,t) \text{ o } N_2(y_2,s)] \geq N((x_1,y_1),t+s)) \ast N((x_2,y_2),t+s))$.

(FN5) The continuity of $N((x,y),\bullet) : [0,\infty) \to [0,1]$ follows from the continuity of $N_1(x,\bullet) :[0.\infty) \to [0,1]$ and the continuity of $N_2(y,\bullet) : [0,\infty) \to [0,1]$ as in the prove of theorem 2.2 . Hence (X×Y,N,o) is a fuzzy normed space .

Proposition 2.4

Let $\{ x_n \}$ be a sequence in the fuzzy normed space $(X,N_1,\ast)$ converges to x in X and $\{y_n\}$ is a sequence in the fuzzy normed space $(Y,N_2,\ast)$ converge to y in Y. If the continuous t-norm o is stronger than $\ast$ then $\{ (x_n,y_n) \}$ is a sequence in the fuzzy normed space (X×Y,N,o) converge to (x,y) in X×Y , where $N = N_1 \text{o} N_2$.

Proof

By Theorem 2.3, (X×Y,N,o) is a fuzzy normed space . Now for each $t > 0$,$s> 0$
$\lim_{n\to\infty} N((x_n,y_n) - (x,y), t+s) = [\lim_{n\to\infty} N_1(x_n - x, t)] \text{o} [\lim_{n\to\infty} N_2(y_n - y, s)] = 1$.
Hence $\{ (x_n,y_n) \}$ converge to (x,y)

Proposition 2.5

Let $\{ x_n \}$ be a Cauchy sequence in the fuzzy normed space $(X,N_1,\ast)$ and $\{y_n\}$ is a Cauchy sequence in the fuzzy normed space $(Y,N_2,\ast)$. If the continuous t-norm o is stronger than the continuous t-norm $\ast$ then $\{ (x_n,y_n) \}$ is a Cauchy in the fuzzy normed space (X×Y,N,o) where $N = N_1 \text{ o } N_2$.

Proof

By Theorem 2.3 , (X×Y,N,o) is a fuzzy normed space . For each $t > 0$,$s> 0$ and p = 1, 2,3,… $\lim_{n\to\infty} N((x_{n+p},y_{n+p}) - (x_n,y_n), t+s) = [\lim_{n\to\infty} N_1(x_{n+p} - x_n, t)]$ $\text{o}[\lim_{n\to\infty} N_2(y_{n+p} - y_n, s)] = 1$. Thus $\{(x_n,y_n)\}$ is a Cauchy sequence in (X×Y,N, o) .

Theorem 2.6

Let (X, N1,$\ast$) and (Y, N2,$\ast$) are complete fuzzy normed spaces . If the continuous t-norm o is stronger than the continuous t-norm $\ast$ then (X×Y, N, o) is a complete fuzzy normed space where $N = N_1 \text{o} N_2$.

Proof

By Theorem 2.3 , (X×Y,N,o) is a fuzzy normed space .Let $\{ (x_n,y_n) \}$ be a Cauchy sequence in X×Y , that is for each $t > 0$, $s> 0$ and p = 1,2,3,…. 1
$= \lim_{n\to\infty} N((x_{n+p},y_{n+p}) - (x_n,y_n), t+s) = [\lim_{n\to\infty} N_1(x_{n+p} - x_n, t)] \text{ o}[\lim_{n\to\infty} N_2(y_{n+p} -$

$y_n, s)$] . Hence $\lim_{n\to\infty} N_1(x_{n+p} - x_n, t)$ = 1 and $\lim_{n\to\infty} N_2(y_{n+p} - y_n, s)$ = 1. Therefore $\{x_n\}$ is a Cauchy sequence in (X, N1,∗) and $\{y_n\}$ is a Cauchy sequence in ($Y,N_2,*$) . But ($X,N_1,*$) and ($Y,N_2, *$) are complete fuzzy normed spaces ,hence there is x in X and y in Y such that for each $t > 0$,$s> 0$ $\lim_{n\to\infty} N_1(x_n - x, t)$ = 1 and $\lim_{n\to\infty} N_2(y_n - y, s)$ = 1 .
So that $\lim_{n\to\infty} N((x_n, y_n) - (x, y), t + s) = [\lim_{n\to\infty} N_1(x_n - x, t)]o[\lim_{n\to\infty} N_2(y_n - y, s)]$= 1.
Hence $\{(x_n,y_n)\}$ converges to (x,y) in X×Y . Thus (X×Y,N,o) is a complete fuzzy normed space .
The prove of the following corollary follows from the proof of Theorem 2.6 and Theorem 2.2. Hence is omitted.

Corollary 2.7

If ($X,N_1,*$) and ($Y,N_2,*$) are complete fuzzy normed spaces then (X×Y,N, o) is a fuzzy normed space .

Theorem 2.8

If (X×Y,N,∗) is a fuzzy normed space then ($X,N_1,*$) and ($Y,N_2,*$) are fuzzy normed spaces by defining $N_1(x,t) = N((x,0),t)$ and $N_2(y,t) = N((0,y),t)$ for all x∈ Xy ∈Y,$t > 0$.

Proof

For each x,x_1, in X and $c \neq 0 \in \mathbb{K}$.
(FN_1) $N_1(x,0) = N((x,0),0) = 0$, for each x∈ X .
(FN_2) For each $t > 0$, $1 = N_1(x,t) = N((x,0),t)$ if and only if x = 0 .
(FN_3) $N_1(cx,t) = N((cx,0),t) = N((x,0),\frac{t}{|c|}) = N_1(x, \frac{t}{|c|})$, for each $t> 0$.
(FN_4) $N_1(x+x_1,s+t) = N((x+x_1,0),,s+t) \geq N((x,0),s) * N((x_1,0),t) = N_1(x,s) * N_1(x_1,t)$.
(FN_5)$N_1(x,\bullet) = N((x,0),\bullet)$ is a continuous function from [0,∞) to [0,1] for all x∈ X.
Thus (X, N1,∗) is a fuzzy normed space. Similarly (Y, N2,∗) is a fuzzy normed space .

Theorem 2.9;

If (X×Y,N,∗) is a complete fuzzy normed space then ($X,N_1, *$) and ($Y,N_2,*$) are complete fuzzy normed spaces where $N_1(x,t) = N((x,0),t)$ and $N_2(y,t) = N((0,y),t)$ for all x ∈ X , y ∈ Y , $t > 0$.

Proof

By Theorem 2.8 , ($X,N_1,*$) and ($Y,N_2,*$) are fuzzy normed spaces .Let $\{ x_n \}$ beCauchy sequence in ($X,N_1,*$) that is for $t > 0$ and p = 1,2,… $\lim_{n\to\infty} N_1(x_{n+p} - x_n, t)$ = 1 .
Now for $t > 0$ and p = 1, 2,…… $\lim_{n\to\infty} N((x_{n+p}, 0) - (x_n, 0), t) = \lim_{n\to\infty} N_1(x_{n+p} - x_n, t)$ = 1 ,this implies that $\{ (x_n,0) \}$ is Cauchy sequence in X×Y. But (X×Y,N,∗) is complete so $\{ (x_n,0) \}$ converge to (x,0) in X×Y, that is for $t > 0$, $\lim_{n\to\infty} N((x_n, 0) - (x, 0), t)$ = 1 . Hence $\lim_{n\to\infty} N_1(x_n - x, t) = \lim_{n\to\infty} N((x_n, 0) - (x, 0), t) = 1$ for each $t > 0$, that is $\{ x_n \}$ converge to x in X . Thus (X, N1,∗) is complete. Similarly we can prove that (Y, N2,∗) is complete.

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