Approximate Solution of Linear Boundary Value Problem Defined on Semi Infinite Interval

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ASTRACT

This paper presents an algorithm for Boundary Value Problems (B.V.P) defined on semi infinite interval approximately. Firstly, the problem is transformed into a new equivalent Volterra integral equation and then Hermite polynomials are used for solving V.I.E approximately with the aid of spectral method. The V.I.E reduces to linear system of algebraic equations with unknown Hermite coefficients'. Comparisons between the exact and approximated results of this method are given via two test examples and accurate results are achieved.

Keywords: Linear Boundary Value Problem, Semi Infinite Interval, Volterra integral equation, Hermite polynomials

الحل التقريبي لمسألة القيم الحدوديه الخطية المعرفة فى الفترة شبه لانهائيه

الخلاصة

يتضمن البحث خوارزميه تقريبيه لمسائل القيم الحدوديه الخطية المعرفة فى الفترة شبه لانهائيه. اولا المسأله يتم تحويلها الى معادله تكامليه مكافئة من نوع فولتيرا وبعد ذلك متعددة حدود من نوع هيرمت استخدمت لحل المعادله التكامليه فولتيرا تقريبيا بواسطه الطريقه الطيفيه. بذلك المعادله التكامليه تُختزل الى نظام خطي من المعادلات الجبريه لمعاملات هيرمت المجهوله. المقارنات بين النتائج المضبوطة والتقريبية أعّطيت لمثالين اختبار وانجزت نتائج دقيقه.

INTRODCTION

During the last few years much progress has been made in the numerical treatment of boundary value problem over infinite intervals. Typically these problems arise very frequently in fluid dynamics, aerodynamic, quantum mechanics, electronic, and other domains of since [1]. As example, heat transfer in the radial flow between parallel circular disks [2], plasma physics [3], and non- Newtonian fluid flows [4].

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In the present paper, we consider the linear Boundary Value Problem of the form:-

$$y''(x) + P(x)\,y(x) = R(x)$$
$$with \quad y(a) = A \quad , \quad y(\infty) = B \quad , \; x \in (a,\infty) \qquad \dots (1)$$

Where P(x) and R(x) are continuous functions, $a \in R,$ A, B constant, before computing the solution, we plummet the infinite interval to a finite but large one, so that a finite.

We introduce a spectral method with hermit polynomials for solving (1), this method consists of change the boundary value problem (1) to Volterra integral equation and expanding y(x) in one of the orthogonal polynomials is hermit polynomials.

HERMITE POLYNOMIALS

The Hermite polynomials $H_n(x)$ are an important set of orthogonal function over the interval $(-\infty,\infty)$ and numerous of the paper used this polynomials foe example, Adzic [13], Yalcinbas & Muge [9], Babusci & Dattoli [16].

The general form of these polynomials is [10]:-

$$H_{n+1}(x) = 2xH_n(x) - 2n\,H_{n-1}(x) \qquad n \ge 1 \qquad \dots (2)$$

Where

$H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$, $H_3(x) = 8x^3 - 12x$, $H_4(x) = 16x^4 - 48x^2 + 12$

PROPERTIES OF HERMITE POLYNOMIALS [14, 11]

1-The general form of the Hermite polynomials of nth degree is given as:-

$$H_n(t) = n!\sum_{j=0}^{N}(-1)^j \frac{2^{n-2j}}{j!(n-2j)!}\, t^{n-2j}$$

Where N=n/2 if n is even and N=(n-1)/2 if n is odd.

Note that this can also be written when n=2, 3,… as:-

$$H_n(t) = \sum_{j=0}^{N}\frac{(-1)^j}{j!}\, n(n-1)...(n-2j+1)(2t)^{n-2j}$$

2- The Hermite polynomials $H_n(x)$ satisfy the Hermite differential equations:-

$$H_n''(t) - 2tH_n'(t) + 2nH_n(t) = 0$$

3- Orthogonality property:-

$$\int_{-\infty}^{\infty} e^{-t^2} \; H_n^2(t)\; dt = 2^n \; n! \sqrt{n} \qquad\qquad n = 0,1,2,\ldots$$

THE SOLUTION OF B.V.P

In this section, we develop a new algorithm consisting of three sections; in the first we change the infinite interval to finite interval, in the second the boundary value problem converts to Volterra integral equation and in the third we find the solution of the Volterra integral using spectral method with Hermite polynomials.

CHANGE THE INFINITE INTERVAL TO FINITE [12]

The boundary condition at infinity is replaced with the semi conditions at a finite value b.

Let us consider a B.V.P (1), we change the second condition when $\begin{matrix} y(\infty)\to B \\ b\to\infty \end{matrix}$ then

$$y(b^N) = B$$

$$b^N = a + (N+1)h \qquad \ldots (3)$$

Let ε be an small arbitrary value, then we us the finite difference method as let

$p_n = p_n(x) \qquad , \qquad r_n = r_n(x)$

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} = p_n y_n + r_n$$

$$y_{n+1} - 2y_n + y_{n-1} = h^2 p_n y_n + h^2 r_n$$

$$y_{n+1} = (2 + h^2 p_n) y_n - y_{n-1} + h^2 r_n \qquad \ldots (4)$$

and by substitute n=1,2,3,…,N+1 in equation (3) we get

$y_2 = (2 + h^2 p_1) y_1 - y_0 + h^2 r_1$

and $y_0 = A$

$$B = (2 + h^2 p_1) y_1 - A + h^2 r_1$$

$$y_1 = \frac{B + A - h^2 r_1}{2 + h^2 p_1}$$

If $\left| y_{(n)}^{N+1} - y_{(n)}^{N} \right| < \varepsilon$ … (5)

is satisfied we stop, else, take then values which obtained in equation (5) and substitute in (4).

CONVERT THE B.V.P TO VOLTERRA INTEGRAL EQUATION

It well known that initial-value and boundary-value problems for differential equations can often be converted into integral equations and there are usually significant advantages to be gained from making use of this conversion [15].

Consider a T.P.B.V.P

$$\begin{aligned} &y''(x) + p(x)y(x) = R(x) \\ &y(a) = A \quad , \quad y(b) = B \end{aligned} \qquad \ldots (6)$$

this equation can be changed to V.I.E to the following:-

$$\int_a^x y''(x)dx = -\int_a^x p(t)\,y(t)dt + \int_a^x R(x)dx$$

$$y'(x) - y'(a) = -\int_a^x p(t)y(t)dt + \int_a^x R(x)dx$$

$$y'(x) = -\int_a^x p(t)y(t)dt + \int_a^x R(x)dx + y'(a)$$

$$let \int_a^x R(x)dx + y'(a) = G(x) + C_1$$

$$\int_a^x y'(x) = -\int_a^x\int_a^t p(t)y(t)dt + \int_a^x G(x)d(x) + C_1 x \Big|_a^x$$

$$let \int_a^x G(x)d(x) + C_1 x \Big|_a^x = F(x)$$

$$by\, rootlip\text{int}\, iz\, therom$$

$$y(x) - y(a) = -\int_a^x p(t)(x-t)\,y(t)\,dt + F(x)$$

$$y(x) = -\int_a^x p(t)(x-t)\,y(t)\,dt + F(x) + A$$

$$let \;\; F(x) + A = M(x)$$

$$y(x) = M(x) - \int_a^x p(t)(x-t)\,y(t)\,dt \qquad \ldots(7)$$

$$M(x) = y(x) + \int_a^x p(t)(x-t)\,y(t)\,dt$$

Which is V.I.E.

Volterra integral equation is widely used for modeling and forecasting in almost all areas of science and engineering [10]. Many researchers used V.I.E in many subject such as Geng [5] used anew method for a Volterra integral equation with weakly singular kernel in the reproducing kernel space, while Masouri [6] presents a numerical expansion- iterative method for solving linear Volterra and Fredholm integral equation of the second kind.

SOLUTION OF V.I.E USING HERMITE POLYNOMIALS

In this section Hermite polynomials accompanied with Spectral method will be used to find the approximate solution of Volterra integral equation (7).

In the Spectral method, the solution is assumed to be a finite linear combination of some set of global analytic basis functions, Hermite polynomials. The integral equation yields then a system of equations for the coefficients.

let $$y_N(x)=\sum_{i=0}^{N} k_i\, H_i(x) \qquad \dots (8)$$

by substituting eq.(7) in (6) we get

$$M(x)=\sum_{i=0}^{N} k_i\, H_i(x)+\int_a^x p(t)(x-t)\sum_{i=0}^{N} k_i\, H_i(t)\,dt \qquad \dots (9)$$

since we have two boundary conditions, we substitute this conditions in (8) to get:-

$$y(a)=k_0H_0(a)+k_1H_1(a)+k_2H_2(a)+\dots+k_NH_N(a)=A$$

$$hence \quad k_0=\frac{A-\sum_{i=1}^{N} k_i\, H_i(a)}{H_0(a)} \qquad \dots (10)$$

$$y(b)=\frac{A-\sum_{i=1}^{N} k_i\, H_i(a)}{H_0(a)}\; H_0(b)+k_1H_1(b)+k_2H_2(b)+\dots+k_NH_N(b)=B$$

$$k_1=\frac{1}{H_1(b)}\left[B-\left(\frac{A-\sum_{i=1}^{N} k_i\, H_i(a)}{H_0(a)}\right)H_0(b)-\sum_{i=2}^{N} k_i\, H_i(b)\right] \qquad \dots (11)$$

then the eq. (9) will be completed as:-

$$y_N(x) = \left(\frac{A - \sum_{i=1}^{N} k_i\, H_i(a)}{H_0(a)}\right) H_0(x) + \frac{1}{H_1(b)}\left[B - \left(\frac{A - \sum_{i=1}^{N} k_i\, H_i(a)}{H_0(a)}\right) H_0(b)\right. \quad \ldots (12)$$
$$\left. - \sum_{i=2}^{N} k_i\, H_i(b)\right] H_1(x) + \sum_{i=2}^{N} k_i H_i(x)$$

and by calculate the value of integration

$$\int_a^x (x-t)\, y_N(t)\, dt = U(x) \quad \ldots (13)$$

Then substitute the (13) into (9) and applied Spectral method with M(x). Hence can be seen as a system of N-2 equations for k_m , this system can be written in matrix from as

$$Ak = G \quad \ldots (14)$$

Gaussian elimination procedure can be used to determine the coefficient k_i's , i=1, 2,,…, N which satisfy eq. (7) (the approximate solution y(x) of eq.(1)).

TEST EXAMPLES

EXAMPLE (1)

Consider the following linear boundary value problem

$$y'' = 9y - 9$$

With the infinite boundary conditions $y(0) = 2$, $y(\infty) = 1$

The exact solution for this problem is $y(x) = e^{-3x} + 1$

Takeing: $h = \frac{b-a}{s}$ and $\varepsilon = 10^{-8}$,

Where s (1) representing the number of chosen points

$$\frac{-y_{n+1} + 2y_n - y_{n-1}}{h^2} = 9y_n - 9$$

$$y_{n+1} = 4.25y_n - y_{n-1} - 2.25 \quad \ldots (15)$$

substituted n=1,2,3,… in (15) to obtain

$y_2 = 4.25y_1 - y_0 - 2.25$ by substituted $y_2 = 0$ we obtain $y_1 = 1$

$y_3 = 4.25y_2 - y_1 - 2.25$ Substituted $y_3 = 0$ to obtain $y_1 = 1.19047619$

But $|1.19047619 - 1| > \varepsilon$, which the equation (5) not satisfied.

so continue to n=5 the second conditions becomes as follows:-

$b^{n+1} = a + (n+1)h$

$b^6 = 2 + (6)0.5$

$b^6 = 5$

The boundary conditions becomes as follows

$y(0) = 2 \quad , \quad y(5) = 1$

Then by changing the B.V.P to V.I.E

$$y'' = 9y - 9 \;\Rightarrow\; y' + 3 = 9\int_0^x y(x)dx - 9x$$

$$y(x) + 3x - 2 = 9\int_0^x (x-t)y(t)dt - \frac{9}{2}x^2$$

$$y(x) = 2 - 3x - \frac{9}{2}x^2 + 9\int_0^x (x-t)y(t)dt \qquad 0 \le x \le 5 \qquad \ldots (16)$$

and by assuming the approximated solution to (16)

$$y_N = \sum_{i=0}^{N} k_i H_i(x)$$

Table (1) presents comparison between the exact and approximated solution for N=6 and N=9 for ten equal space points depending on h=0.6 which depends on least square error.

EXAMPLE (2)

Consider the following linear boundary value problem

$y'' - y = 0$

With the infinite boundary condition

$y(0) = 1 \;, \quad y(\infty) = 0$

the exact solution for this problem is $y(x) = e^{-x}$

we take $\varepsilon = 10^{-10}$ by solving it in the same way on example (1) we obtain n=19 the second condition will be as follows:-

$b^{(19+1)} = a + (19+1)\ 0.5$

$b^{20} = 0 + (20)0.5$

$b^{20} = 10$

so the boundary conditions becomes y(0)=1 , y(10)=0

now by changing the B.V.P to V.I.E

$$y(x) = 1 - x + \int_0^x (x-t)y(t)dt \qquad 0 \le x \le 10$$

and by assuming the approximate solution

$$y_N = \sum_{i=0}^{N} k_i H_i(x)$$

Table (2) presents comparison between the exact and approximated solution for N=8 and N=11 for ten equal space points depending on h=1 which depends on least square error.

CONCLUTIONS

Hermite Spectral method was used for the approximate solution of linear boundary value problems define on semi infinite intervals. The algorithm consists of three stages, In the first stage, the infinite interval was changed into finite one, in the second stage the boundary value problem was converted into Volterra integral equation, in the third stage, the obtained volterra integral equation was solved with the aid of Hermite polynomials. Two examples have been taken and comparison was made between the exact solution and the numerical solutions using different values of N. the results showed that the proposed method is efficient.

REFRENCE

[1]. Maleki, M. M. Tavassolil, " A Non classical Collection Method For Solving Two-Point Boundary Value Problems Over Infinite Intervals", Australian Journal of Basic and Applied Sciences, 5(9):1045-1050,2011.

[2]. "Computational Method in Engineering Boundary Value Problems", T.Y. Na, Academic Press, New York (1979).

[3]. Agrwal, R.P. D.O. Ragan, " Infinite Interval Problems Modeling Phenomena Which Arise in the Theory of Plasma and Electrical Potential Theory", Stud.Appl.Math, 111(2003), 339-358.

[4]. Agrwal, R.P. D.O. Ragan, " Infinite Interval Problems Arising in Non-Linear Mechanics and Non-Newtonian Fluid Flows", Internal. J. Non-Linear Mech. 38(2003), 1368-1376.

[5]. Geng, F. F. Shen, "Solving a Volterra Integral Equation with Weakly Singular Kernel in the Reproducing Kernel Space", Mathematical Sciences, Vol.4.No.2 (2010), 159-170.

[6]. Masouri, Z. " Numerical Expansion-Iterative Method for Solving Second Kind Volterra and Fredholm Integral Equations Using Block-Pulse Functions", Advancrd Computational Techniques in Electromagnetic, Vol. 2012, Year 2012, Article ID acte-00108, 7 Pages.

[7]. Palamara,A. " Product Integration for Volterra Integral Equations of the Second Kind with Weakly Singular Kernel", Mathematics of Computation, Vol. 65, No. 215, July (1996), 1201-1212.

[8]. Kareem,R. "Approximated Treatment of Higher-Order Linear Volterra Integral Differential Equation", M. Sc. Thesis, Applied Science Department, University of Technology, 2003.

[9]. Yalcinbas,S . M. Aynigul," Hermite Series Solutions of Linear Fredholm Integral Equations", Mathematical and Computational Applications, Vol. 16, No. 2, PP 497-506,2011.

[10]. Jawad,A. "Expansion Methods for Solving Linear Integral Equations with Multiple Time lags Using B-Spline and Orthogonal Functions", Eng. & Tech. Journal, Vol. 2, No. 9,2011.

[11]. Poularikas, A. "Hermite Polynomials", The Handbook of Formulas and Tables for Signal Processing, Boca Raton, CRC Press LLC, 1999.

[12]. Al-Bayati, A. Y. K. I. Ibraheem, "A Modified Wavelet Algorithm to Solve B.V.P with An Infinite Number of Boundary Conditions", Int. J. Open Problems Compt. Math. Vol. 4, No.2, June 2011.

[13]. Adzic,N. " Recurrence Relations For The Hermite Solution of an Ordinary Differential Equation with Polynomial Coefficient", Univ. u Novom Sadu, Zb. Rad. Prirod-Mat.Fak, Ser. Mat.22,1(1992),167-177.

[14]. Gulsu,M. H. Yalman, "A New Hermite Collection Method For Solving Differential Difference Equations", Applications and Applied Mathematics: An International Journal, Vol. 6, Issue 11(June 2011), PP.1856-1869.

[15]. Saberi,J. O. Reza, "Numerical Solution of Two Dimensional Volterra Integral Equations by Spectral Galerkin Method", Journal of applied Mathematics & Bioinformatics, Vol. 1, No.2, 2011, 159-174.

[16]. Babusci,D. G. Dattoli, M. Quattrmini," On Integrals Involving Hermite Polynomials", Applied Mathematics Letters 25 (2012) 1157–1160, http:// www.ivsl.org.

Table (1).

X	Y-Exact	Y-Approximate	
		N=6	N=9
0	2	2	2
0.5	1.223130	1.223231	1.223123
1	1.049787	1.048988	1.049777
1.5	1.011109	1.011114	1.011198
2	1.002479	1.002367	1.002469
2.5	1.000553	1.000548	1.000552
3	1.000123	1.000215	1.000122
3.5	1.000028	1.000191	1.000019
4	1.000006	1.000011	1.000005
4.5	1.000001	1.000002	1.000001
5	1	1	1
L.S.E		0.000001	$0.8\ e^{-7}$

Table (2).

X	Y-Exact	Y-Approximate	
		N=8	N=11
0	1	1	1
1	0.367878	0.367778	0.367887
2	0.135335	0.135256	0.135312
3	0.049787	0.049689	0.049777
4	0.018316	0.018265	0.018397
5	0.006737	0.006684	0.006726
6	0.002478	0.002475	0.002469

7	0.000911	0.000912	0.000911
8	0.000335	0.000326	0.000329
9	0.000123	0.000115	0.000122
10	0	0	0
L.S.E		$0.3\ e^{-7}$	$0.7\ e^{-9}$