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Similarity of Solutions of Non Conformally Perfect Fluid Models of Embedding Class One

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ABSTRACT

The Study of similarity Transformation Method has played a remarkable role of dealing with many immediate theoretical and practical applications in our life problems, such as physical and engineering problems [6].This method find the solution for the complicated problems which contains coupled higher non linear partial differential equations. By using this method we can seek a solution to spherical fluid spheres models. In this paper we found a new solution and analyzed it with respect to its physical acceptability conditions [5].Here we deals with a perfect fluid distribution described a metric in the non conformally flat form ,it has been a subject of interest. The science of hydrodynamics deals with motion of fluids which is defined as a collection of molecules [2]. All liquids are compressible to a slight extent, but for many purposes it is simpler to consider the liquid as being incompressible. However the pressure is the same in all directions for a perfect fluid. [8].

Keywords: Similarity Solutions; Einstein's Field Equations, Perfect Fluid.

الحلول المتماثلة لموديلات الموائع التامة الغير متطابقة مغمورة في فئة رقم (1)

الخلاصة

دراسة طريقة التحويلات المتماثلة لعبت دور واضح في التعامل مع العديد من المسائل الحياتية النظرية وتطبيقاتها الواقعية مثل المسائل الفيزياوية والهندسية .فهي تبحث حل أصعب المسائل التي تحتوي على معادلات تفاضلية جزئية عالية اللاخطية .هذه الطريقة تقوم على أيجاد الحل للنماذج الكروية للموائع.

في هذا البحث وجدنا حل جديد وقمنا بتحليله طبقا لشروطه الفيزياوية المقبولة ،هنا نتعامل مع توزيع الموائع التامة والتي معيارها هو بصيغة مسطح غير متطابق والذي يمثل موضوع أهتمامنا.أن علم قوة الموائع يتعامل مع حركة الموائع والتي تعرف على أنها مجاميع من الجزيئات .فالموائع كلها بأستثناء الماء تعتبر قابلة للأنضغاط أنضغاطا خفيفا.لكن لعدة أغراض من السهولة أن نتعامل مع الموائع على أنها غير قابلة للأنضغاط .على أية حال الضغط سيكون نفسه في كل الأنجاهات للمائع التام.

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INTRODUCTION

There exists two types of perfect fluid distributions, one for which Wyle conformal tensor vanishes and the other one for which is non-vanishing [3]. In other words, the former are said to be conformally flat and the latter are called non-conformally flat perfect fluid distribution [7].

Barnes, 1974 [1] has shown that the non-conformally case divided into two parts, one with vanishing acceleration vector and the other which it is non-zero [12].the solution has been found for those belonging to the former case .However, for latter case, very few solutions are available, which include the solution due to Kohler and Choa [6],the only static non-conformal solution .Besides this, Gupta et al [4] has also contributed towards it by finding the most general stiff fluids of a class of solutions in 5-flat form.

Gupta and Jasim [5], have found some accelerating fluid spheres of embedding class one with non-vanishing conformal curvature tensor. Jasim [8] have been found a new class of solution describing fluid distributions with non-zero acceleration.

In the present paper, we have again tackled the problem of non-conformal perfect fluid distributions of class one by considering the powerful technique, which is a similarity transformation method (STM) to such process of getting a new solution.

Before us discussing our method on a relativistic model we will apply this method on a physical problem as a one dimensional heat equation as follows

APPLICATIONS OF STM ON A ONE DIMENSIONAL HEAT EQUATION

Let us consider the one dimensional heat equation

$$u_{xx} - u_t = 0 \qquad \text{... (1)}$$

We consider the group of transformations as

$$\left\{ \begin{aligned} u^1 &= u^1(x,t,u;\varepsilon) \\ x^1 &= x^1(x,t,u;\varepsilon) \\ t^1 &= t^1(x,t,u:\varepsilon) \end{aligned} \right\} \qquad \text{... (2)}$$

Put (1) and its accompanying boundary conditions invariant. Then this invariance implies that $v = u^1$ which satisfying the equation

$$v_{x^1x^1} - v_{t^1} = 0 \qquad \text{... (3)}$$

Iff $u_{xx} - u_t = 0$ and in terms of the new variables the original boundary condition must be satisfied if $u = \phi(x,t)$ is a solution to (1) then we can conclude that $v = \phi(x^1, t^1)$ is a solution to (3) but by using the transformation (2) we get that (1) remaining invariant so we get a solution to (3) we must demand that

$$u^1(x,t,\phi(x,t);\varepsilon) = \phi(x^1, t^1) \qquad \text{... (4)}$$

We consider the $o(\varepsilon)$ terms in the expansions of u^1, x^1, t^1:

$$\left\{\begin{aligned} u^1 &= u + \varepsilon U(x,t,u) + o(\varepsilon^2) \\ x^1 &= x + \varepsilon X(x,t,u) + o(\varepsilon^2) \\ t^1 &= t + \varepsilon T(x,t,u) + o(\varepsilon^2) \end{aligned}\right\}$$

Where

$$U = \left(\frac{\partial u^1}{\partial \varepsilon}\right)_{\varepsilon=0} \quad , \quad X = \left(\frac{\partial x^1}{\partial \varepsilon}\right)_{\varepsilon=0} \quad , \quad T = \left(\frac{\partial t^1}{\partial \varepsilon}\right)_{\varepsilon=0}$$

Expanding (4) and equating $o(\varepsilon)$ terms ,then replacing $\phi(x,t)$ by u we get:

$$U(x,t,u) = X(x,t,u)\frac{\partial u}{\partial x} + T(x,t,u)\frac{\partial u}{\partial t} \qquad \dots (5)$$

Eq. (5) is the general partial differential equation of an invariant surface.

Then we can conclude the characteristic equations corresponding to (5) by using Lagrange subsidiary equations, which are

$$\frac{du}{U(x,t,u)} = \frac{dx}{X(x,t,u)} = \frac{dt}{T(x,t,u)} \qquad \dots (6)$$

In principle (6) is solvable, thus we obtain

$$u = u(x,t,\eta : F(\eta)) \qquad \dots (7)$$

Where the dependence of u on x and t is known explicitly, η is a similarity variable which will be found from solving (6) which will be independent of u if $\frac{X}{T} = f(x,t)$, and the dependence of u on η involves some arbitrary function $F(\eta)$,substituting (7) into (1) we reduce it to 2^{nd} order ordinary differential equation with independent variable η & dependent variable $F(\eta)$.

Obviously our problem is to find a largest possible class of infinitesimals U, X, T for (1) In general, for linear equations.

$$\left.\begin{cases}\dfrac{X(x,t,u)}{T(x,t,u)} = u\, f(x,t) \\ \dfrac{U(x,t,u)}{T(x,t,u)} = u\, f(x,t) + f(x,t)\end{cases}\right\} \qquad \dots (8)$$

Now we find the $o(\varepsilon)$ terms in the expansion of $u^1{}_{x^1x^1} - u^1{}_{t^1}$ as:

$$\begin{aligned} u^1{}_{t^1} &= u^1_t + u^1_x\, x_{t^1} \\ &= \left[u_t + \varepsilon(U_t + U_u u_t)\right]\left[1 - \varepsilon T_t - \varepsilon T_u u_t\right] - \varepsilon u_x \left[X_t + X_u u_t\right] + o(\varepsilon^2) \\ &= u_t + \varepsilon\left[-X_u u_t u_x - T_u u_t u_t + (U_u - T_t) u_t - X_t u_x + U_t\right] + o(\varepsilon^2) \end{aligned}$$

In a same way we can find

$$u^1_{x^1} = u_x + \varepsilon[-T_u u_t u_x - X_u u_x u_x + (U_u - X_x) u_x - T_x u_t + U_x] + o(\varepsilon^2)$$

Thus for invariance we can conclude eq. (1) as follows:

$$\begin{aligned} u^1_{x^1x^1} &= (u^1_{x^1})_x\, x_{x^1} + (u^1_{x^1})_t\, t_{x^1} = u_{xx} + \varepsilon[-T_{uu} u_x u_x u_t - X_{uu} u_x u_x u_x - 2T_u u_x u_{tx} \\ &\quad - (3X_u + 2T_{xu}) u_x u_t - T_u u_t u_t + (U_{uu} - 2X_{ux}) u_x u_x - 2T_x u_{tx} + (U_u - 2X_x - T_{xx}) u_t \\ &\quad + (2U_{xu} - X_{xx}) u_x + U_{xx}] + o(\varepsilon^2) = u^1_{t^1} \end{aligned}$$

Now we want to take our relativistic model and find its solution by applying STM technique.

3. **Einstein field equations**

An embedding class one space time can be written as

$$ds^2 = -\left(dz^1\right)^2 - \left(dz^2\right)^2 - \left(dz^3\right)^2 + \left(dz^4\right)^2 - \left(dz^5\right)^2 \qquad \dots (9)$$

Where $z^1 = r\sin\theta\cos\phi$, $z^2 = r\sin\theta\sin\phi$, $z^3 = r\cos\theta$, $z^4 = t$,

$$z^5 = u\left(r,t\right) \qquad \dots (10)$$

Then we get the metric

$$ds^2 = -dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) + dt^2 - du^2 \qquad \dots (11)$$

The Einstein's field equations for the perfect fluid distributions can be expressed as [4]

$$8\pi T^i_j = -R^i_j + \frac{1}{2}\delta^i_j R = 8\pi\left[\left(p+\rho\right)v^i v_j - \delta^i_j p\right] \qquad \dots (12)$$

Where $v^i v_j = 1$ and $\rho,\ p$ and v^i are energy density, pressure and flow vector respectively [10].

The Einstein field equations (12) so that the metric (11) can be furnished as below

$$8\pi T^1_1 = \frac{u'^2}{r^2 P} - \frac{2u'}{rP^2}\left[\left(1+u'^2\right)\ddot{u} - \dot{u}u'\dot{u}'\right] = 8\pi\left(\rho+p\right)v^1 v_1 - 8\pi p \qquad \dots (13)$$

$$8\pi T^2_2 = 8\pi T^3_3 = -\frac{1}{P^2}\left(\ddot{u}u'' - \dot{u}'^2\right) - \frac{u'}{rP^2}\left[\left(1+u'^2\right)\ddot{u} - 2D\dot{u}u'\dot{u}' - \left(1-\dot{u}^2\right)u''\right] \dots (14)$$
$$= -8\pi p,$$

$$8\pi T^4_4 = \frac{u'^2}{r^2 P} + \frac{2u'}{rP^2}\left[\left(1-\dot{u}^2\right)u'' + \dot{u}u'\dot{u}'\right] = 8\pi\left(\rho+p\right)v^4 v_4 - 8\pi p, \qquad \dots (15)$$

$$8\pi T^4_1 = \frac{2u'}{rP^2}\left[\left(1+u'^2\right)\dot{u}' - \dot{u}u'u''\right] = 8\pi\left(\rho+p\right)v^4 v_1, \qquad \dots (16)$$

$$8\pi T^1_4 = -\frac{2u'}{rP^2}\left[\left(1-\dot{u}^2\right)\dot{u}' + \dot{u}u'\ddot{u}\right] = 8\pi\left(\rho+p\right)v^1 v_4, \qquad \dots (17)$$

$$v^1 v_1 + v^4 v_4 = 1,\quad v^2 = v_2 = v^3 = v_3 = 0, \qquad \dots (18)$$

Where $P = 1 - \dot{u}^2 + u'^2$

However $\rho,\ p$ are expressible in terms of T^i_j , s as follows

$$p = -T^2_2 \qquad ,\rho = T^1_1 + T^4_4 - T^2_2$$

Elimination of $\rho,\ p$ and v^i among the above (13-18) equations one gets the following condition

$$(3F - 8\pi\rho)(F - 8\pi p) = 0 \qquad \dots (19)$$

Where $F = \dfrac{2(\ddot{u}u'' - \dot{u}'^2)}{p^2}$

The vanishing of the first factor in (19) implies the vanishing of the conformal curvature tensor and the corresponding fluid distribution will be conformally flat however the vanishing of the 2^{nd} factor corresponds to the fluid distributions with non vanishing conformal tensor in the later case.

So that the perfect fluid may be non conformally flat which can be expressed as:

$$\left(\ddot{u}u''-\dot{u}'^2\right)-\frac{u'}{r}\left[\left(1+u'^2\right)\ddot{u}-2\dot{u}u'\dot{u}'-\left(1-\dot{u}^2\right)u''\right]-\frac{u'^2}{r^2}\left[1-\dot{u}^2+u'^2\right]=0 \ldots (20)$$

Associated expressions for pressure and density are given by

$$8\pi\,\rho=\frac{3u'^2}{r^2P}, \qquad \ldots (21)$$

and

$$8\pi\,p=\frac{2\left(\ddot{u}u''-\dot{u}'^2\right)}{P^2}-\frac{u'^2}{r^2P} \qquad \ldots (22)$$

SIMILARITY SOLUTIONS OF THE FIELD EQUATION

Similarity method of solving partial differential equations is extremely powerful approach to get the solutions even for highly nonlinear equation in the process of getting the groups transformations under which the given equation is to be invariant; we come across the Lie symmetries of equation (20) as follows [6]

1. Check for invariance under the transformation.

$$\begin{matrix} x^*=Ax \\ y^*=By \end{matrix}, \quad 0<A\,,\,B<\infty$$

2. Admit the transformations for one parameter group.

$$Uf=\xi(x,y)\frac{\partial f}{\partial x}+\eta(x,y)\frac{\partial f}{\partial y}$$

3.Write the differential equation in the form.

$$H(x,y,y',y'')=y''-w(x,y,y')=0$$

Here H denotes the perfect fluid distribution which is non conformally given by equation (20) Hence

$$\text{H}=\left(\ddot{u}u''-\dot{u}'^2\right)-\frac{u'}{r}\left[\left(1+u'^2\right)\ddot{u}-2\dot{u}u'\dot{u}'-\left(1-\dot{u}^2\right)u''\right]-\frac{u'^2}{r^2}\left[1-\dot{u}^2+u'^2\right]=0 \quad \ldots (23)$$

4. Then we can calculate the following

$$\frac{\partial H}{\partial x} = -w_x(x,y,y')$$
$$\frac{\partial H}{\partial y} = -w_y(x,y,y')$$
$$\frac{\partial H}{\partial y'} = -w_{y'}(x,y,y')$$
$$\frac{\partial H}{\partial y''} = 1$$

5. Now use the criteria $U'' = 0$,

We find that $Uf = \xi(x,y)\dfrac{\partial f}{\partial x} + \eta(x,y)\dfrac{\partial f}{\partial y} + \eta'(x,y)\dfrac{\partial f}{\partial y'} + \eta''(x,y)\dfrac{\partial f}{\partial y''} = 0$

Here we will discuss the similarity method in details by the derivation of the dependent variable η as follows:

$$[\eta_{rr}]\frac{\partial H}{\partial u_{rr}} + [\eta_{rt}]\frac{\partial H}{\partial u_{rt}} + [\eta_{tt}]\frac{\partial H}{\partial u_{tt}} + [\eta_r]\frac{\partial H}{\partial u_r} + [\eta_t]\frac{\partial H}{\partial u_t} + \eta\frac{\partial H}{\partial u} + \xi\frac{\partial H}{\partial r} + \tau\frac{\partial H}{\partial t} = 0 \cdots \text{(24)}$$

Given that

$\eta, \xi, \tau, [\eta_r], [\eta_t], [\eta_{rr}], [\eta_{rt}], [\eta_{tt}]$ are the infinitesimals with the associated derivatives of H are given by the following expressions:

$$[\eta_r] = \frac{\partial \eta}{\partial r} + (\frac{\partial \eta}{\partial u} - \frac{\partial \xi}{\partial r})\theta_r - \frac{\partial \tau}{\partial r}\theta_t - \frac{\partial \xi}{\partial u}\theta_r^2 - \frac{\partial \tau}{\partial u}\theta_r\theta_t$$

$$[\eta_t] = \frac{\partial \eta}{\partial t} + (\frac{\partial \eta}{\partial u} - \frac{\partial \tau}{\partial t})\theta_t - \frac{\partial \xi}{\partial t}\theta_r - \frac{\partial \tau}{\partial u}\theta_t^2 - \frac{\partial \xi}{\partial u}\theta_t\theta_r$$

$$\begin{aligned}[\eta_{rr}] = {} & \frac{\partial^2 \eta}{\partial r^2} + (2\frac{\partial^2 \eta}{\partial r \partial u} - \frac{\partial^2 \xi}{\partial r^2})\theta_r - \frac{\partial^2 \tau}{\partial r^2}\theta_t + [\frac{\partial^2 \eta}{\partial u^2} - 2\frac{\partial^2 \xi}{\partial r \partial u}]\theta_r^2 - 2\frac{\partial^2 \tau}{\partial r \partial u}\theta_r\theta_t \\ & - \frac{\partial^2 \xi}{\partial u^2}\theta_r^3 - \frac{\partial^2 \tau}{\partial u^2}\theta_r^2\theta_t + (\frac{\partial \eta}{\partial u} - 2\frac{\partial \xi}{\partial r})\theta_{rr} - 2\frac{\partial \tau}{\partial r}\theta_{rt} - 3\frac{\partial \xi}{\partial u}\theta_{rr}\theta_r \\ & - \frac{\partial \tau}{\partial u}\theta_{rr}\theta_t - 2\frac{\partial \tau}{\partial u}\theta_{rt}\theta_r\end{aligned}$$

$$[\eta_{tt}] = \frac{\partial^2\eta}{\partial t^2} + (2\frac{\partial^2\eta}{\partial t\partial u} - \frac{\partial^2\tau}{\partial t^2})\theta_t - \frac{\partial^2\xi}{\partial t^2}\theta_r + [\frac{\partial^2\eta}{\partial u^2} - 2\frac{\partial^2\tau}{\partial t\partial u}]\theta_t^2 - 2\frac{\partial^2\xi}{\partial t\partial u}\theta_r\theta_t$$
$$-\frac{\partial^2\tau}{\partial u^2}\theta_t^3 - \frac{\partial^2\xi}{\partial u^2}\theta_t^2\theta_r + (\frac{\partial\eta}{\partial u} - 2\frac{\partial\tau}{\partial t})\theta_{tt} - 2\frac{\partial\xi}{\partial t}\theta_{rt} - 3\frac{\partial\tau}{\partial u}\theta_{tt}\theta_t$$
$$-\frac{\partial\xi}{\partial u}\theta_{tt}\,\theta_r - 2\frac{\partial\xi}{\partial u}\theta_{rt}\theta_t$$

We obtain determining equations of the group by equating the coefficients of θ and its derivatives to zero [11].

Then by using the symbolic computation we can get the following formulas:

$$\eta' = \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2 ,$$

and

$$\eta'' = \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\xi_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + (\eta_y - 2\xi_x)y'' - 3\xi_y\ y'y''$$

Then use the criteria which we will apply later.

6. Set the coefficient of power of y are equal to zero, then solve for $\xi(x,y)$ *and* $\eta(x,y)$

7. Use the characteristic differential equation and solve the characteristic equation it should be reduced to the form $\frac{dv}{du} = \phi(u,v)$, which is first order differential equation and can be solved by any other method or may be again the STM can be applied. Thus, equation (20) has been transformed after some simplification with use of the above procedure and help of Jasim, 1999 [9], to get the symmetry infinitesimals as

$$\xi = a\,r\,,\ \ \eta = a\,u + b\,,\ \tau = a\,t + c \qquad \ldots(25)$$

Where a, b, c are three arbitrary parameters?

Forms of the solutions of the equation (20) can be obtained by solving the following with reference to (25) we can write the following Lagrange subsidiary equations as

$$\frac{dr}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta} \qquad \ldots (26)$$

Accordingly, many cases may arise which immediately suggests the form of u as

$$u = \alpha\,t + f\left(r\right) \qquad \text{Where} \quad \alpha = \frac{b}{c} \qquad \ldots (27)$$

On inserting the value of u from equation (26) into equation (20), we have

$$r\left(1-\alpha^2\right) f'' = -f'\left(1-\alpha^2 + f'^2\right) \qquad \ldots (28)$$

Which represent the general solution of equation (20?)

CONCLUSIONS

STM have played a remarkable role in dealing with highly nonlinear equations. The solution with non zero acceleration are very rare. Anyhow new solution have been obtained & analyzed subject to the energy conditions, it has large density and the pressure at the initial stages. Moreover the ratio of energy density to pressure remains in acceptable limit so the solution so obtained is physically acceptable.

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