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# Properties of Fuzzy Closed Linear Operator

***Abstract***- *In this paper we recall the definition of fuzzy norm of a fuzzy bounded linear operator and the fuzzy convergence of sequence of fuzzy bounded linear operators in order to prove the uniform fuzzy bounded theorem and fuzzy open mapping theorem. The definition of fuzzy closed linear operators on fuzzy normed spaces is introduced in order to prove the fuzzy closed graph theorem.*

***Keywords***- *Fuzzy bounded linear operator, fuzzy norm of a fuzzy bounded operator, Fuzzy open mapping, Fuzzy closed graph, Fuzzy closed operator.*

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## 1. Introduction

Zadeh in 1965[1] was the first author who find the theory of fuzzy set. When Katsaras in 1984 [2] studying the notion of fuzzy topological vector spaces he was the first researcher who find the notion of fuzzy norm on a linear vector space. A fuzzy metric space was found by Kaleva and Seikkala in 1984 [3]. The type of fuzzy norm on a linear space was found by Felbin in 1992 [4] where the corresponding fuzzy metric is of Kaleva and Seikkala type. Another type of fuzzy metric space was found by Kramosil and Michalek in [5]. The type of fuzzy norm on a linear space was found by Cheng and Mordeson in 1994 [6] so that the corresponding fuzzy metric is of Kramosil and Michalek type. A finite dimensional fuzzy normed linear spaces was studied by Bag and Samanta in 2003 [7]. Some results on fuzzy complete fuzzy normed spaces was studied by Saadati and Vaezpour in 2005 [8]. Fuzzy bounded linear operators on a fuzzy normed space was studied by Bag and Samanta in 2005 [9]. The fixed point theorems on fuzzy normed linear spaces of Cheng and Mordeson type was proved by Bag and Samanta in 2006, 2007 [10], [11]. The fuzzy normed linear space and its fuzzy topological structure of Cheng and Mordeson type was studied by Sadeqi and Kia in 2009 [12]. Properties of fuzzy continuous mapping on a fuzzy normed linear spaces of Cheng and Mordeson type was studied by Nadaban in 2015 [13]. The definition of the fuzzy norm of a fuzzy bounded linear operator was introduced by Kider and Kadhum in 2017 [14]. Fuzzy functional analysis is developed by the concepts of fuzzy norm and a large number of researches by different authors have been published for reference please see [ 16, 17, 18, 19,20, 21, 22, 23].The structure of this paper is as follows:

In section two we recall the definition of fuzzy normed space duo to Cheng and Mordeson [6] and we recall some basic definitions and properties of this space that we will need it later in this paper. In section three the definition of three types of fuzzy convergence sequence of operators is recalled in order to prove the uniform fuzzy bounded theorem. Also the fuzzy open mapping theorem is proved and the fuzzy closed graph theorem is proved after defining the fuzzy closed linear operator.

## 2. Properties of Fuzzy normed space

In this section we recall basic properties of fuzzy normed space

**Definition 2.1: [1]**

Suppose that U is any set, a fuzzy set $\widetilde{A}$ in U is equipped with a membership function, $\mu_{\widetilde{A}}(u)$: U→[0,1]. Then $\widetilde{A}$ is represented by $\widetilde{A}=\{(u,\mu_{\widetilde{A}}(u)): u\in U, 0\leq\mu_{\widetilde{A}}(u)\leq 1\}$.

**Definition 2.2: [25]**

Let $\otimes:[0,1]\times[0,1]\to[0,1]$ be a binary operation then $\otimes$ is called a continuous **t -norm** (or **triangular norm**) if for all $\alpha,\beta,\gamma,\delta\in[0,1]$ it has the following properties

(1)$\alpha\otimes\beta=\beta\otimes\alpha$, (2)$\alpha\otimes 1=\alpha$,
(3)$(\alpha\otimes\beta)\otimes\gamma=\alpha\otimes(\beta\otimes\gamma)$
(4)If $\alpha\leq\beta$ and $\gamma\leq\delta$ then $\alpha\otimes\gamma\leq\beta\otimes\delta$

**Remark 2.3: [22]**

(1)If $\alpha>\beta$ then there is $\gamma$ such that $\alpha\otimes\gamma\geq\beta$

 

(2)There is $\delta$ such that $\delta\otimes\delta \geq \sigma$ where $\alpha, \beta, \gamma, \delta, \sigma \in [0,1]$

**Definition 2.4: [8]**

The triple $(V, L, \otimes)$ is said to be a **fuzzy normed space** if V is a vector space over the field $\mathbb{F}$, $\otimes$ is a t-norm and $L: V \times [0, \infty) \to [0,1]$ is a fuzzy set has the following properties for all a, b$\in$ V and $\alpha, \beta > 0$.

1-$L(a, \alpha) > 0$

2-$L(a, \alpha) = 1 \Leftrightarrow a = 0$

3-$L(ca, \alpha) = L\left(a, \frac{\alpha}{|c|}\right)$ for all $c \neq 0 \in \mathbb{F}$

4-$L(a, \alpha) \otimes L(b, \beta) \leq L(a + b, \alpha + \beta)$

5-$L(a, .): [0, \infty) \to [0,1]$ is continuous function of $\alpha$.

6-$\lim_{\alpha\to\infty} L(a, \alpha) = 1$

**Remark 2.5: [24]**

Assume that $(V, L, \otimes)$ is a fuzzy normed space and let $a \in V, t > 0, 0 < q < 1$. If $L(a,t) > (1 - q)$ then there is s with $0 < s < t$ such that $L(a, s) > (1 - q)$.

**Definition 2.6: [6]**

Suppose that $(V, L, \otimes)$ is a fuzzy normed space. Put

$FB(a, p, t) = \{b \in V : L(a - b, t) > (1 - p)\}$

$FB[a, p, t] = \{b \in V : L(a - b, t) \geq (1 - p)\}$

Then $FB(a, p, t)$ and $FB[a, p, t]$ is called **open and closed fuzzy ball** with the center a $\in$Vand radius p, with $p > 0$.

**Lemma 2.7: [8]**

Suppose that $(V, L, \otimes)$ is a fuzzy normed space then $L(x - y, t) = L(y - x, t)$ for all $x, y \in V$ and $t > 0$

**Definition 2.8: [6]**

Assume that $(V, L, \otimes)$ is a fuzzy normed space. W $\subseteq$V is called **fuzzy bounded** if we can find $t > 0$ and $0 < q < 1$ such that $L(w, t) > (1 - q)$ for each $w \in W$.

**Definition 2.9: [8]**

A sequence $(v_n)$ in a fuzzy normed space $(V, L, \otimes)$ is called **converges to** $v \in V$ if for each $q > 0$ and $t > 0$ we can find N with $L[v_n - v, t] > (1 - q)$ for all $n \geq N$. Or in other word $\lim_{n\to\infty} v_n = v$ or simply represented by $v_n \to v$, v is known the limit of $(v_n)$ or $\lim_{n\to\infty} L[v_n - v, t] = 1$.

**Definition 2.10: [8]**

A sequence $(v_n)$ in a fuzzy normed space $(V, L, \otimes)$ is said to be a **Cauchy sequence** if for all $0 < q < 1$, $t > 0$ there is a number N with $L[v_m - v_n, t] > (1 - q)$ for all $m, n \geq N$.

**Definition 2.11: [4]**

Suppose that $(V, L, \otimes)$ is a fuzzy normed space and let W be a subset of V. Then the **closure of W** is written by $\overline{W}$ or $CL(W)$ and which is $\overline{W} = \bigcap\{W \subseteq B: B \text{ is closed in } V\}$.

**Lemma 2.12: [14]**

Assume that $(V, L, \otimes)$ is a fuzzy normed space and suppose that W is a subset of V. Then $w \in \overline{W}$ if and only if there is a sequence $(w_n)$ in W with $(w_n)$ converges to w.

**Definition 2.13: [14]**

Suppose that $(V, L, \otimes)$ is a fuzzy normed space and $W \subseteq V$. Then W is called **dense** in V when $\overline{W} = V$.

**Theorem 2.14: [14]**

Suppose that $(V, L, \otimes)$ is a fuzzy normed space and assume that W is a subset of V. Then W is dense in V if and only if for every $u \in V$ there is $w \in W$ such that

$L[u - w, t] > (1 - \varepsilon)$ for some $0 < \varepsilon < 1$ and $t > 0$.

**Definition 2.15: [10]**

A fuzzy normed space $(V, L, \otimes)$ is said to be **complete** if every Cauchy sequence in V converges to a point in V.

**Definition 2.16: [8]**

Suppose that $(V, L_V, \otimes)$ and $(W, L_W, \odot)$ are two fuzzy normed spaces .The operator $S: V \to W$ is said to be **fuzzy continuous at $v_0$** $\in$ V if for all $t > 0$ and for all $0 < \alpha < 1$ there is s[depends on t, $\alpha$ and $v_0$] and there is $\beta$ [depends on t, $\alpha$ and $v_0$] with, $L_V[v - v_0, s] > (1 - p)$ we have $L_W[S(v) - S(v_0), t] > (1 - \alpha)$ for all v $\in$V.

**Definition 2.17: [14]**

Let $(V, L_V, \otimes)$ and $(U, L_U, \odot)$ be two fuzzy normed spaces. The operator T:D(T)$\to$ U is said to be **fuzzy bounded** if there exists r, $0 < r < 1$ such that

$L_U(Tv, t) \geq (1 - r) \otimes L_V(v, t)$ for each $v \in D(T) \subseteq V$ and $t > 0$ where D(T) is the domain of T.

**Theorem 2.18: [14]**

Suppose that $(V, L_V, \otimes)$ and $(U, L_U, \odot)$ are two fuzzy normed spaces. The operator $S: D(S) \to U$ is fuzzy bounded if and only if S(A) is fuzzy bounded for every fuzzy bounded subset A of D(S).

Put FB(V,U) ={S:V$\to$U, S is a fuzzy bounded operator} when $(V, L_V, \otimes)$ and $(U, L_U, \odot)$ are two fuzzy normed spaces [14].

**Lemma 2.19: [14]**

Let $(V, L_V, \otimes)$ be a fuzzy normed space. If A and B are fuzzy bounded subset of V then A+B and $\alpha$A are fuzzy bounded for any $\alpha \neq 0 \in \mathbb{F}$.

**Theorem 2.20: [14]**

Suppose that $(V, L_V, \otimes)$ and $(U, L_U, \odot)$ are two fuzzy normed spaces. Define $\mathbf{L(T, t) = \inf_{x\in D(T)} L_Y(Tx, t)}$ for all $T \in FB(V, U)$ and $t > 0$ then $(FB(V, U), L, *)$ is fuzzy normed space.

**Theorem 2.21: [14]**

Suppose that $(V, L_V, \otimes)$ and $(U, L_U, \odot)$ are two fuzzy normed spaces with $S: D(S) \to U$ is a linear operator where $D(S) \subseteq V$. Then S is fuzzy bounded if and only if S is fuzzy continuous.

**Corollary 2.22: [14]**

Let $(V, L_V, \otimes)$ and $(U, L_U, \odot)$ be two fuzzy normed spaces. Assume that $T: D(T) \to U$ is a linear operator where $D(T) \subseteq V$. Then T is a fuzzy continuous if T is a fuzzy continuous at $x \in D(T)$.

**Theorem 2.23: [14]**

Let $(V, L_V, \otimes)$ and $(U, L_U, \odot)$ be two fuzzy normed spaces. If U is fuzzy complete then $FB(V, U)$ is fuzzy complete.

**Definition 2.24: [14]**

A linear functional f from a fuzzy normed space $(V, L_V, \otimes)$ into the fuzzy normed space $(F, L_F, \odot)$ is said to be **fuzzy bounded** if there exists r, $0 < r < 1$ such that $L_F[f(x), t] \geq (1 - r) \otimes L_X[x, t]$ for all $x \in D(f)$ and $t > 0$. Furthermore, the fuzzy norm of f is $L(f, t) = \inf L_F(f(x), t)$ and $L_F(f(x), t) \geq L(f, t) \otimes L_X(x, t)$.

**Definition 2.25: [14]**

Suppose that $(V, L_V, \otimes)$ is a fuzzy normed space. Then the vector space $FB(V, \mathbb{F}) = \{f: V \to \mathbb{F}$, f is fuzzy bounded linear functional $\}$ with a fuzzy norm defined by

$\mathbf{L(f, t) = \inf L_F(f(x), t)}$ form a fuzzy normed space which is called the fuzzy dual space of V.

## 3. Fuzzy Convergence of Sequence of Operators and Functional

**Definition 3.1: [24]**

A sequence $(v_n)$ in a fuzzy normed space $(V, L_V, \otimes)$ is said to **weakly fuzzy convergent** if we can find $v \in V$ with every $h \in FB(V, \mathbb{R})$ $\lim_{n \to \infty} h(v_n) = h(v)$. This is written $v_n \to^w v$ the element v is said to be the weak limit of $(v_n)$ and $(v_n)$ is said to be fuzzy converges weakly to v.

**Theorem 3.2: [24]**

Suppose that $(v_n)$ is in the fuzzy normed space $(V, L_V, \otimes)$.

1. If $v_n \to v$ then $v_n \to^w v$.
2. If $v_n \to^w v$ and dimension of V is finite then $v_n \to v$.

**Definition3.3: [24]**

Suppose that $(V, L_V, \otimes)$ and $(U, L_U, \odot)$ are two fuzzy normed spaces. A sequence $(T_n)$ operators $T_n \in FB(V, U)$ is said to be

**1. Uniform operator fuzzy convergent** if there is $T: V \to U$ with $L[T_n - T, t] \to 1$ for any $t > 0$ and $n \geq N$.

**2. Strong operator fuzzy convergent** if there is $T: V \to U$ with $L_U[T_n v - Tv, t] \to 1$ for every $t > 0$, $v \in V$ and $n \geq N$.

**3. Weak operator fuzzy convergent** if there is $T: V \to U$ with $L_R[f(T_n v) - f(Tv), t]$ for every $t > 0$, $f \in FB(U, R)$ and $n \geq N$.

Respectively, T is called uniform strong and weak operator limit of $(T_n)$

**Definition3.5: [24]**

Let $(V, L_V, \otimes)$ be a fuzzy normed space. A sequence $(h_n)$ of functional $h_n \in FB(V, \mathbb{R})$ is called

1) **Strong fuzzy converges** if there is $h \in FB(V, \mathbb{R})$ with $L[h_n - h, t] \to 1$ for all $t > 0$ this written $h_n \to h$

2) **Weak fuzzy converges** if there is $h \in FB(V,R)$ with $h_n(v) \to h(v)$ for every $v \in V$ written by $\lim_{n \to \infty} h_n(v) = h(v)$.

**Definition 3.6:**

A subset E of a fuzzy normed space $(V, L, \otimes)$ is said to be

1- **Rare** (or **no where dense**) in V if its closure $\bar{E}$ has no interior points

2-**Meager** (or of the **first category**) in V if E is the union of countable many sets each of which is rare in V

3-**Nonmeager** (or of the second category) in V if E is not meager in V

**Theorem [Fuzzy Baire's theorem] 3.7:**

Let $(V, L_V, \otimes)$ be a complete fuzzy normed space such that $V \neq \emptyset$, then it is non meager in itself.

**Proof:**

We must prove that when $V \neq \emptyset$ is complete and $V = \cup_{k=1}^{\infty} V_k$ ($V_k$ closed) then there is one $V_k$ contains open subset which is not equal to $\emptyset$.

Let V be meager in itself that is

$V = \cup_{k=1}^{\infty} M_k \ldots(1)$

such that every $M_k$ is rare in V, we will find a Cauchy sequence $(p_k)$ whose limit is p is in no $M_k$ this contradict the equation (1) by assumption $M_1$ is rare in V so that $\overline{M_1}$ does not contains a nonempty open set but V does. It follows $\overline{M_1} \neq V$. Thus $\overline{M_1^c} = V - \overline{M_1} \neq \emptyset$ and open. Now choose a point $q_1$ in $\overline{M_1^c}$ and an open fuzzy ball about it, say

$B_1 = FB(q_1, \varepsilon_1, t) \subset \overline{M_1^c}$. By assumption $M_2$ is rare in V so that $\overline{M_2}$ does not contains a nonempty open set hence $FB\left(q_1, \frac{\varepsilon_1}{2}, t\right) \not\subset \overline{M_2}$ this implies that $\overline{M_2^c} \cap FB\left(q_1, \frac{\varepsilon_1}{2}, t\right)$ is not empty and open, let $B_2 = FB(q_2, \varepsilon_2, t) \subseteq \overline{M_2} \cap FB(q_1, \varepsilon_1, t)$ by induction we obtain a sequence of open fuzzy balls $B_k = FB(q_k, \varepsilon_k, t)$, such that

$B_k \cap M_k = \emptyset$ and $B_{k+1} \subset FB\left(q_k, \frac{\varepsilon_k}{2}, t\right) \subset B_k$. Let $\varepsilon_k < 2^{-k}$ and $(1 - \varepsilon_k) > 1 - 2^{-k}$ the sequence $(q_m)$ of the centers is a Cauchy and converges say $q_m \longrightarrow q \in V$ because V is complete as we assume. Now for all m and n with $n > m$ with $B_n \subset FB\left(q_m, \frac{\varepsilon_m}{2}, t\right)$ so that

$L_V(q_m - q, t)$

$$\geq L_V\left(q_m - q_n, \frac{t}{2}\right) \otimes L_V\left(q_n - q, \frac{t}{2}\right)$$
$$\geq \left(1 - \frac{\varepsilon_m}{2}\right) \otimes L_V\left(q_n - q, \frac{t}{2}\right)$$
$$= \left(1 - \frac{\varepsilon_m}{2}\right) \otimes 1 = \left(1 - \frac{\varepsilon_m}{2}\right)$$

as $n \to \infty$

Hence $q \in B_M$ for every m since $B_m \subset \overline{M_m^c}$, so $q \notin M_m$ for any m, so that $q \notin \cup_{m=1}^{\infty} M_m = V$. This contradicts $q \in V$.

We need the following theorem in the next results

**Uniform fuzzy Bounded Theorem 3.8:**

Let $(T_n)$ be sequence in $FB(V, U)$ where $(V, L_V, \otimes)$ is a complete fuzzy normed space and $(U, L_U, \odot)$ is a fuzzy normed space such that $(L_U[T_n v, t])$ is fuzzy bounded for every $v \in V$ with $L_U[T_n v, t] \geq (1 - c_v)$ ….(2) n=1, 2,… where $0 < c_v < 1$.

Then $(L[T_n, t])$ is fuzzy bounded so there is $0 < c < 1$ with

$L[T_n, t] \geq (1 - c)$….(3) $n \in \mathbb{N}$.

**Proof:**

For $m \in (0, 1)$ let $V_m$={ $v \in V$: $L_U[T_n v, t] \geq (1 - m)$ } for all $v \in \overline{V}_m$ there is a sequence $(v_j)$ in $V_m$ converges to v. This means that for every fixed $n$ we have $L_U[T_n v_j, t] \geq (1 - m)$ and obtain $L_U[T_n v, t] \geq (1 - m)$ since $T_n$ is fuzzy continuous. Hence $v \in V_m$ and $V_m$ is closed now by (2) each $v \in V$ belongs to some $V_m$. Hence $V = \cup_{m=1}^{\infty} V_m$. Since V is complete at this point Fuzzy Baire's theorem implies that some $V_m$ contains an open fuzzy ball say,

$B_0 = FB(v_0, r, t) \subset V_{m_0}$…(4)

Let $v \in V$ be nonzero, let $z = v_0 + v$ with $L_V[z - v_0, t] > \left(1 - \frac{r}{2}\right)$ so that $L_V[z - v_0, t] > (1 - r)$ which implies $z \in B_0$. Now by (4) and from definition of $V_{m_0}$ we have

$L_U[T_n z, \frac{t}{2}] \geq (1 - m_0)$ for all n. Also

$L_U\left[T_n v_0, \frac{t}{2}\right] \geq (1 - m_0)$ since $v_0 \in B_0$.

Now for all $v \in V$

$$L_U[T_n v, t] = L_U[T_n(z - v_0), t]$$
$$\geq L_U\left[T_n z, \frac{t}{2}\right] \odot L_U\left[T_n v_0, \frac{t}{2}\right]$$
$$\geq (1 - m_0) \odot (1 - m_0).$$

Hence for all n,

$L[T_n, t] = \inf L_U[T_n v, t] \geq (1 - c)$ where for some $(1 - m_0) \odot (1 - m_0) \geq (1 - c)$ for some $0 < c < 1$.

The following lemma is needed later

**Lemma 3.9:**

Let $T_n \in FB(V, U)$ where $(V, L_V, \otimes)$ is a complete fuzzy normed space and $(U, L_U, \odot)$ is a fuzzy normed space if $(T_n)$ is a strong operator fuzzy convergent with limit T then $T \in FB(V, U)$.

**Proof**

Suppose that T is linear since $T_n v \to Tv$ for every $v \in V$ and $T_n$ is linear. The sequence $(T_n v)$ is fuzzy bounded for any $v \in V$. Now since V is complete so $(L[T_n, t])$ is fuzzy bounded by uniform fuzzy bounded Theorem 3.8, say $L[T_n, t] \geq (1 - c)$ for all n and for some $0 < c < 1$. From this it follows that $L_U[T_n v, t] \geq L[T, t] \otimes L_V[v, t]$. This implies $L_U[Tv, t] \geq (1 - c) \otimes L_V[v, t]$ hence $T \in FB(V, U)$.

**Definition 3.10:**

Let $(V, L_V, \otimes)$ be a fuzzy normed space then a subset D of V is called a **fuzzy total set** if $\overline{\text{spanD}} = V$.

**Theorem 3.11:**

A sequence $(T_n) \in FB(V, U)$ where $(V, L_V, \otimes)$ and $(U, L_U, \odot)$ are complete fuzzy normed spaces is strong operator fuzzy convergent if and only if

(i) $(L[T_n, t])$ is fuzzy bounded sequence.

(ii) $(T_n d)$ is a Cauchy sequence in U for any $d \in E$ where E is a fuzzy total subset of V.

**Proof:**

Since $T_n v \to Tv$ for every $v \in V$ then $(i)$ is satisfied from Theorem 3.8 since V is complete, also (ii) is satisfied.

Conversely suppose that (i) and (ii) holds let $L\left[T_n, \frac{t}{3}\right] \geq (1 - c)$ for all n and t >0

$0 < c < 1$. We consider any $v \in V$ and prove that $(T_n v)$ converges strongly in U. Let $0 < \varepsilon < 1$ and $t > 0$ be given since spanE is dense in V we have $y \in$ spanE with $L_V\left[v - y, \frac{t}{3}\right] > (1 - \varepsilon)$ for some $0 < \varepsilon < 1$ the sequence $(T_n y)$ is Cauchy by (ii) hence for given $0 < r < 1$ and $t > 0$ there is a number N such that $L_U\left[T_n y - T_m y, \frac{t}{3}\right] > (1 - r)$.

Now for n, m ≥N we get

$$L_U[T_n v - T_m v, t] \geq L_U\left[T_n v - T_n y, \frac{t}{3}\right] \odot L_U\left[T_n y - T_m y, \frac{t}{3}\right] \odot L_U\left[T_m y - T_m v, \frac{t}{3}\right]$$
$$\geq L\left[T_n, \frac{t}{3}\right] \otimes L_V\left[v - y, \frac{t}{3}\right] \odot L_U\left[T_n y - T_m y, \frac{t}{3}\right] \odot L\left[T_m, \frac{t}{3}\right] \otimes L_V\left[v - y, \frac{t}{3}\right]$$
$$> (1 - c) \otimes (1 - \varepsilon) \odot (1 - r) \odot (1 - c) \otimes (1 - \varepsilon).$$

Now we can find $0 < (1 - p) < 1$ so that

$(1 - c) \otimes (1 - \varepsilon) \odot (1 - r) \odot (1 - c) \otimes (1 - \varepsilon) > (1 - p)$. Therefore

$L_U[T_n v - T_m v, t] > (1 - p)$ for every n, m ≥N. Hence $(T_n v)$ is Cauchy in U but U is complete $(T_n v)$ convergence in U. Since $v \in V$ was arbitrary this proves strong operator fuzzy convergence of $(T_n)$.

**Definition 3.12:**
Suppose that $(V, L_V, \otimes)$ and $(U, L_U, \odot)$ are two fuzzy normed spaces. The operator $T: D(T) \to U$ where $D(T) \subseteq V$ is said to be a **fuzzy open** if for any open set E in D(T)
T(E) is open set in U.
The following lemma is the key of proving the next main results

**Lemma 3.13:**
Suppose that $(V, L_V, \otimes)$and $(U, L_U, \odot)$ are two complete fuzzy normed spaces and let $T: V \to U$ be a fuzzy bounded linear surjective operator. Then $T(B_0)$ the image of the open fuzzy ball $B_0 = FB(0,1,t) \subset V$ contains an open fuzzy ball a bout $0 \in U$.

**Proof:**
First step we will prove that the closure of the image of the open fuzzy ball $B_1 = FB\left(0, \frac{1}{2}, t\right)$ contains an open fuzzy ball $B^*$. For a subset E of V we know that $\alpha E = \{v \in V: v = \alpha z, z \in E\}$ where $\alpha \neq 0 \in F$ and $E + w = \{v \in V: v = z + w, z \in E\}$. We consider the fuzzy open ball $B_k = FB\left(0, \frac{1}{2^k}, t\right)$. Hence $V = \bigcup_{k=1}^{\infty} B_k$ since T is surjective and linear $U = T(V) = T(\bigcup_{k=1}^{\infty} B_k)$ $= \bigcup_{k=1}^{\infty} T(B_k) = \bigcup_{k=1}^{\infty} \overline{T(B_k)}$ since U is complete it is nonmeager in itself by Theorem 3.7. Hence we concluded that $\overline{T(B_1)}$ must contain some open fuzzy ball, say $B^* = FB(y_0, \varepsilon, t) \subset \overline{T(B_1)}$. It follows that

$B^* - y_0 = FB(0, \varepsilon, t) \subset \overline{T(B_1)} - y_0$ … (5).

In the second step we will show that $\overline{T(B_n)}$ contains an open fuzzy ball $D_n$ about $0 \in U$ where $B_n = FB(0, 2^{-n}, t) \subset V$. To prove $\overline{T(B_1)} - y_0 \subset \overline{T(B_0)}$ let $y \in \overline{T(B_1)} - y_0$. Then $y + y_0 \in \overline{T(B_1)}$ we know that $y_0 \in \overline{T(B_1)}$. Then by lemma 2.12 there are $u_n = Tw_n \in T(B_1)$ such that $u_n \to y + y_0$, $v_n = Tz_n \in T(B_1)$ such that $v_n \to y_0$ since $w_n, z_n \in B_2$ it follows that

$$L_V[w_n - z_n, t] \geq L_V\left[w_n, \frac{t}{2}\right] \otimes L_V\left[z_n, \frac{t}{2}\right] \geq \left(1 - \frac{1}{2}\right) \otimes \left(1 - \frac{1}{2}\right)$$

so that $w_n - z_n \in B_0$. From

$T(w_n - z_n) = Tw_n - Tz_n = u_n - v_n \to y$

we say that $y \in \overline{T(B_0)}$. So $\overline{T(B_1)} - y_0 \subset \overline{T(B_0)}$. Thus from (5) we get $B^* - y_0 = FB(0, \varepsilon, t) \subset \overline{T(B_0)}$ from this we obtain

$V_n = FB\left(0, \frac{\varepsilon}{2^n}, t\right) \subset \overline{T(B_n)}$…(6)

where $B_n = FB(0, 2^{-n}, t) \subset V$ with $B_0 \supset B_1 \supset B_2 \supset \cdots$ so $\overline{T(B_0)} \supset \overline{T(B_1)} \supset \cdots \supset \overline{T(B_n)} \supset \cdots$ that is $\overline{T(B_n)} \subset \overline{T(B_0)}$. In the final step we prove that $T(B_0)$ contains an open fuzzy ball a bout $0 \in U$ that is we must prove that $V_1 = FB\left(0, \frac{\varepsilon}{2}, t\right) \subset T(B_0)$ let $y \in V_1$ from (6) with $n = 1$ we have $V_1 \subset \overline{T(B_1)}$ hence $y \in \overline{T(B_1)}$ there must be $v \in T(B_1)$ such that $L_U[y - v, t] > \left(1 - \frac{\varepsilon}{4}\right)$. Now $v \in T(B_1)$ implies $v = T(x_1)$ for some $x_1 \in B_1$. Hence $L_U[(y - Tx_1), t] > \left(1 - \frac{\epsilon}{4}\right)$ from this and (6) with $n = 2$ we see that $y - Tx_1 \in V_2 \subset \overline{T(B_2)}$. As before there is $x_2 \in B_2$ such that $L_U[(y - Tx_1) - Tx_2, t] > \left(1 - \frac{\epsilon}{8}\right)$ hence
$y - Tx_1 - Tx_2 \in V_3 \subset \overline{T(B_3)}$ and so on. In the nth step we can find $x_n \in B_n$ such that

$L_U[y - \sum_{k=1}^{n} Tx_k, t] > \left(1 - \frac{\varepsilon}{2^{n+1}}\right)$ (n = 1,2,3, ...) …(7)

Let $z_n = \sum_{i=1}^{n} x_i$ since $x_k \in B_k$ we have $L_V[x_k, t] > \left(1 - \frac{1}{2^k}\right)$. For m>n

$$L_V[z_n - z_m, t] > \left(1 - \frac{1}{2^{m+1}}\right) \otimes \left(1 - \frac{1}{2^{m+2}}\right) \otimes \ldots \otimes \left(1 - \frac{1}{2^n}\right)$$

Put $\left(1 - \frac{1}{2^{m+1}}\right) \otimes \left(1 - \frac{1}{2^{m+2}}\right) \otimes \ldots \otimes \left(1 - \frac{1}{2^n}\right) > (1 - r)$ for some $0 < r < 1$ hence
$L_V[z_n - z_m, t] > (1 - r)$ for all $n, m > N$ for some positive number N. Thus $(z_n)$ is Cauchy so $z_n \to x$ because V is complete. Also $x \in B_0$ since $x = \sum_{k=1}^{\infty} x_k$ and
$B_0 \supset B_1 \supset B_2 \supset \cdots$ and T is fuzzy continuous, $Tz_n \to Tx$ and (7) show that T(x)= y. Hence $y \in T(B_0)$.
The following result is the main result

**Theorem [fuzzy open mapping theorem] 3.14:**
Suppose that $(V, L_V, \otimes)$ and $(U, L_U, \odot)$ are two complete fuzzy normed spaces and let $S: V \to U$ be a surjective fuzzy bounded linear operator then S is a fuzzy open mapping. Hence if S is bijective then $S^{-1}$ is fuzzy continuous hence fuzzy bounded.

**Proof:**
We will prove that if E⊂V is open then S(E) is open in U that is we will show that for every u= S(v) ∈ S(E) the set S(E) contains an open fuzzy ball about u= S(v) ∈ S(E). Let u= S(v) ∈ S(E ) but E is open so it contains an open fuzzy ball with center v. Hence E−v contain an open fuzzy ball with center 0 let the radius of the fuzzy ball be r, $0 < r < 1$. Since $FB(0, 1, t) \supset FB(0, r, t)$ for each $0 < r < 1$. Hence E−v contains the open fuzzy ball $FB(0, 1, t)$. Now Lemma 3.13 implies that $S(E - v) = S(E) - S(v)$ contains an open fuzzy ball a bout 0 and so dose $S(E) - S(v)$. Thus S(E) contains an open fuzzy ball about u= S(v). Since u ∈ S(E) was arbitrary S(E) is open in U.
Now if $S^{-1}: U \to V$ exists it is continuous since S is fuzzy open. Because $S^{-1}$ is linear then by Theorem 2.21 it is fuzzy bounded.

**Definition 3.15:**
Suppose that $(V, L_V, \otimes)$ and $(U, L_U, \otimes)$ are two fuzzy normed spaces and let $S: D(S) \to U$ be a linear operator with D(S) is a subset of V then S is said to be a **fuzzy closed linear operator** if its graph $G(S) = \{(v, u): v \in D(S), u = S(v)\}$ is closed in the fuzzy normed space $(V \times U, L, \otimes)$ where $L[(v, u), t] = L_V[v, t] \otimes L_U[u, t]$

**Theorem [fuzzy closed graph theorem] 3.16:**
Suppose that $(V, L_V, \otimes)$ and $(U, L_U, \otimes)$ are complete fuzzy normed spaces and let $S: D(S) \to U$ be a closed linear operator with D(S) is a subset of V. Then S is fuzzy bounded if D(S) is closed in V.

**Proof:**
First $(V \times U, L, *)$ is complete fuzzy normed space [24]. By assumption $G(S)$ is closed in $V \times U$ and $D(S)$ is closed in V. Hence $G(S)$ and $D(S)$ are complete [24]. Now define $P: G(S) \to D(S)$ by $P[(v, S(v))] = v$ then P is linear and P is fuzzy bounded since

$$L[P(v, S(v)), t] = L_V[v, t] \geq L_V[v, t] \otimes L_U[Sv, t] = L[(v, S(v)), t]$$

and P is bijective so $P^{-1}$ exists where $P^{-1}: D(S) \to G(S)$ defined by $P^{-1}[v] = (v, S(v))$ since $G(S)$ and $D(S)$ are complete we can apply Theorem 3.14 and see $P^{-1}$ is fuzzy bounded, say

$L[(v, S(v))] \geq (1 - r) \otimes L_V[v, t]$ for some $0 < r < 1$ and for all $v \in D(S)$.

Hence S is fuzzy bounded because

$$L_U[Sv, t] \geq L_U[Sv, t] \otimes L_V[v, t] = L[(v, S(v)), t] \geq (1 - r) \otimes L_V[v, t] \text{ for all } v \in D(S).$$

We need the following lemma in the next result.

**Lemma 3.17:**
Suppose that $(V, L_V, \otimes)$ and $(U, L_U, \otimes)$ are two fuzzy normed spaces and let $S: D(S) \to U$ be a linear operator with D(S) is a subset of V then S is fuzzy closed if and only if when $v_n \to v$ where $v_n \in D(S)$ and $Sv_n \to u$ then $v \in D(S)$ and $S(v) = u$ where $u \in U$.

**Proof:**
$G(S)$ is closed if and only if $z = (v, u) \in \overline{G(S)}$ implies $z \in G(S)$, Now from Theorem 3.16 we see that $z \in \overline{G(S)}$ if and only if there are $z_n = (v_n, S(v_n)) \in G(S)$ such that $z_n \to z$ hence $v_n \to v, Sv_n \to u$ and $z = (v, u) \in G(S)$ if and only if $v \in D(S)$ and $u = S(v)$.

**Theorem 3.18:**
Suppose that $(V, L_V, \otimes)$ and $(U, L_U, \otimes)$ are two fuzzy normed spaces and let $S: D(S) \to U$ be fuzzy bounded linear operator with D(S) is a subset of V then:

1-If $D(S)$ is closed in V then S is closed

2-If S is closed and U is complete then $D(S)$ is closed in V.

**Proof:**
1-Suppose that $(v_n) \in D(S)$ and $v_n \to v$ such that $Sv_n \to Sv$ then $v \in \overline{D(S)} = D(S)$ since $D(S)$ is closed and $Sv_n \to Sv$ since S is fuzzy continuous. Hence S is fuzzy closed by Lemma 3.17.

2-Let $v \in \overline{D(S)}$ there is $(v_n)$ in $D(S)$ with $v_n \to v$ since S is fuzzy bounded

$$L_U[Sv_n - Sv_m, t] = L_U[S(v_n - v_m), t] \geq L[S, t] \otimes L_V[v_n - v_m, t].$$

This show that $(Sv_n)$ is Cauchy so $Sv_n \to u \in U$ since U is complete because S is fuzzy closed $v \in D(S)$ and $Sv = u$. Hence $D(S)$ is closed because $v \in \overline{D(S)}$ was arbitrary.

## 4.Conclusion

The main goal of this paper is to prove the fuzzy open mapping theorem and to introduce the definition of fuzzy closed linear operators on fuzzy normed spaces in order to prove the fuzzy closed graph theorem and to investigate some basic properties of this type of operators. We have tried here to translate the results in functional analysis to fuzzy context and we succeeded in this situations.

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