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Weak Convergence of Two Iteration Schemes in Banach Spaces

Abstract- In this paper, we established weak convergence theorems by using appropriate conditions for approximating common fixed points and equivalence between the convergence of the Picard-Mann iteration scheme and Liu et al iteration scheme in Banach spaces. As well as, numerical examples are given to show that Picard-Mann is faster than Liu et al iteration schemes.

Keywords- Banach space, weak convergence, common fixed points.

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1. Introduction and Preliminaries

Let B be a nonempty subset of a Banach space M . A map T on B is called nonexpansive map if $\|Ta - Tb\| \leq \|a - b\|$ for all $a, b \in B$. It is called quasi-nonexpansive map [1] if $\|Ta - b\| \leq \|a - p\|$ for all $a \in B$ and for all $p \in F(T)$, denote by $F(T)$ the set of all fixed point of T .

In 2008, a new condition for maps, called condition (C) was introduced by Suzuki [2], which is stronger than quasi-nonexpansive and weaker than nonexpansive, and given some results about fixed point for map satisfying condition (C) . Dhompongsa et al [3] and Phuengrattana [4] studied fixed point theorems for a map satisfying condition (C) . Weak convergence theorem for a map satisfying condition (C) in uniformly convex Banach space are proved by Kahn and Suzuki [5]. Recently, Garcial-Falset et al [6] introduced two new generalization of condition (C) , called condition (E_λ) , condition (C_λ) and studied the existence of fixed points and also their asymptotic behavior. For approximating common fixed point of two maps, Takahashi and Tamura [7] studied the following Ishikawa iteration scheme for two nonexpansive maps.

$$a \in B$$

$$a_{n+1} = (1 - \alpha_n)a_n + \alpha_n T b_n$$

$$b_n = (1 - \beta_n)a_n + \beta_n T a_n$$

for all $n \in N$, (α_n) and (β_n) in $[0,1]$.

The aim of this paper is to study weak convergence of the Picard-Mann iteration scheme, Liu et al

iteration scheme for approximating common fixed point of generalized nonexpansive and quasi-nonexpansive maps and give some corollaries. Throughout this paper, M will be a uniformly convex Banach space and B a nonempty closed convex subset of M . $F(T, S)$ denotes the set of common fixed point of the maps S and T .

The Picard-Mann [8] iteration scheme for two maps through the sequence (a_n) is defined by:

$$\begin{aligned} a_{n+1} &= S b_n \\ b_n &= (1 - \alpha_n)a_n + \alpha_n T a_n, \forall n \geq 0 \end{aligned} \quad (1)$$

where $(\alpha_n) \in (0,1)$.

The Liu et al [9] iteration scheme for two maps through the sequence (z_n) is defined by:

$$\begin{aligned} z_{n+1} &= (1 - \alpha_n)S z_n + \alpha_n T u_n \\ u_n &= (1 - \beta_n)S z_n + \beta_n T z_n, \forall n \geq 0 \end{aligned} \quad (2)$$

where (α_n) and $(\beta_n) \in [0,1]$.

If $S = I$ is called Ishikawa iteration scheme.

Definition (1.1): A Banach space M is called satisfying:

1-Opial's condition [10] if for any sequence (a_n) in M , is weakly convergent to a implies that

$$\liminf_{n \rightarrow \infty} \|a_n - a\| < \liminf_{n \rightarrow \infty} \|a_n - b\|$$

for all $b \in M$ with $a \neq b$.

2-Kadec-Klee property [11] if for every sequence (a_n) in M converging weakly to (a) together with $\|a_n\|$ converging strongly to $\|a\|$ imply that (a_n) converges strongly to a point $a \in M$.

Defintion (1.2)[12]: A map $T: B \rightarrow M$ is said to be generalized nonexpansive map if there are

nonnegative constants δ, μ and ω with $\delta + 2\mu + 2\omega \leq 1$ such that $\forall a, b \in B$

$$\|Ta - Tb\| \leq \delta \|a - b\| + \mu \left\{ \begin{array}{l} \|a - Ta\| \\ + \|b - Tb\| \end{array} \right\} + \omega \left\{ \begin{array}{l} \|a - Tb\| \\ + \|b - Ta\| \end{array} \right\}$$

Definition (1.3)[13]: A map $T: B \rightarrow B$ is said to satisfying:

1- Condition (C) if $\frac{1}{2} \|a - Ta\| \leq \|a - b\|$ yields $\|Ta - Tb\| \leq \|a - b\|, \forall a, b \in B$.

2-Condition (C $_{\lambda}$) if $\lambda \|a - Ta\| \leq \|a - b\|$ yields $\|Ta - Tb\| \leq \|a - b\|, \forall a, b \in B$ and $\lambda \in (0, 1)$.

3-Condition (E $_{\lambda}$) if $\|a - Tb\| \leq \lambda \|a - Tb\| + \|a - b\|, \forall a, b \in B$ and $\lambda \geq 1$.

Remark (1.4): A map $T: B \rightarrow M$ satisfy

1-Condition(C $_{\lambda}$) and T has fixed point, then T is quasi-nonexpansive, but the inverse is false[2].

2-Condition(E $_{\lambda}$) and T has fixed point, then T is quasi-nonexpansive, but the inverse is false[6].

Definition (1.5)[14]: A map $T: B \rightarrow M$ is said to be demiclosed with respect to $b \in M$ if for any sequence (a_n) in B , (a_n) converges weakly to a and $T(a_n)$ converges strongly to b . Then $a \in B$ and $T(a) = b$. If $(I - T)$ is demiclosed i.e if (a_n) converges weakly to a in B and $(I - T)$ converges strongly to 0. Then $(I - T)(a) = 0$.

Definition (1.6)[15]: Let M be a Banach space, M is called uniformly convex if for any $\epsilon > 0$ there is $\zeta > 0$ such that $\forall a, b \in M$ with $\|a\| = \|b\| = 1$ and $\|a - b\| \geq \epsilon, \|a + b\| \leq 2(1 - \zeta)$ holds. Every uniformly convex Banach space is reflexive. - The modulus of convexity of M is defined by

$$\zeta_M(\epsilon) = \inf \left\{ 1 - \frac{\|a + b\|}{2}; \|a\| = \|b\| \leq 1, \|a - b\| \geq \epsilon, \forall 0 < \epsilon \leq 2 \right\}$$

M is uniformly convex if $\zeta_M(0) = 0$ and $\zeta_M(\epsilon) \geq 0, \forall 0 < \epsilon \leq 2$.

Theorem (1.7)[15]: let M be a uniformly convex Banach space then the modulus of convexity is increasing function.

Remark (1.8)[16]: If $\zeta(0) = 0$ and has the properties :

$\zeta(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow +0$.

$\zeta: [0, 2] \rightarrow [0, 1]$ is strictly monotone increasing function and surjective

Then $\eta: [0, 1] \rightarrow [0, 2]$ is called strictly monotone increasing function of ζ .

Theorem (1.9)[15]: let M be a uniformly convex Banach space. Then for any r and ϵ with $r \geq \epsilon > 0$ and elements $a, b \in M$ such that $\|a\| \leq r, \|b\| \leq r, \|a - b\| \geq \epsilon, \exists \delta = \delta\left(\frac{\epsilon}{r}\right) > 0$ such that

$$\left\| \frac{a + b}{2} \right\| \leq r \left[1 - \delta \left(\frac{\epsilon}{r} \right) \right].$$

Proposition (1.10)[16]: Let B be a closed convex set in a Banach space M . If (a_n) converges weakly to a for some sequence (a_n) in M , then $a \in M$.

Lemma (1.11)[17]: Let $(\mu)_{n=0}^{\infty}$ and $(\rho)_{n=0}^{\infty}$ be nonnegative real sequences satisfying the inequality:

$$\mu_{n+1} \leq (1 - \sigma_n)\mu_n + \rho_n$$

where $\sigma_n \in (0, 1), \forall n \geq n_0, \sum_{n=1}^{\infty} \sigma_n = \infty$ and $\frac{\rho_n}{\sigma_n} \rightarrow 0$ as $n \rightarrow \infty$. Then $\lim_{n \rightarrow \infty} \mu_n = 0$.

Lemma (1.12)[13]: Let M be a uniformly convex Banach space and $0 < L \leq t_n \leq K < 1, \forall n \in N$. Suppose that (a_n) and (b_n) are two sequences of M such that:

$$\begin{aligned} \lim_{n \rightarrow \infty} \|a_n\| \leq m, \lim_{n \rightarrow \infty} \|b_n\| &\leq m \text{ and } \lim_{n \rightarrow \infty} \left\| \frac{t_n a_n + (1 - t_n) b_n}{1 - t_n} \right\| \\ &= m \end{aligned}$$

hold for some $m \geq 0$. Then $\lim_{n \rightarrow \infty} \|a_n - b_n\| = 0$.

Lemma (1.13)[18]: Let B be a nonempty convex subset of a uniformly convex Banach space. Then there is a strictly increasing continuous function $f: [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ such that for each Lipschitz map $T: B \rightarrow B$ with Lipschitz constant K :

$$\begin{aligned} \|tTx + (1 - t)Ty - T(tx + (1 - t)y)\| &\leq Kf^{-1} \left(\frac{\|x - y\|}{K} \right) \\ &\leq Kf^{-1} \left(\frac{1}{K} \|Tx - Ty\| \right) \end{aligned}$$

$\forall x, y \in B$ and $\forall t \in [0, 1]$.

Lemma (1.14)[18]: Let M be a uniformly convex Banach space such that its dual M^* satisfies the Kadec-Klee property. Assume that (a_n) bounded sequence in M such that

$$\lim_{n \rightarrow \infty} \|ta_n + (1 - t)p_1 - p_2\| \text{ exists } \forall t \in [0, 1] \text{ and } p_1, p_2 \in W_w(a_n), \text{ then } p_1 = p_2.$$

2.The Main Results

Proposition (2.1): Let B be a closed convex bounded of uniformly convex Banach space, $T: B \rightarrow M$ is a generalized nonexpansive map and $a_0, a_1 \in B, a_0 \neq a_1 \forall t \in [0,1], a_1 = ta_0 + (1-t)a_1$. If $\forall \epsilon > 0, \exists a(\epsilon) > 0$ such that $\|Ta_0 - a_0\| \leq \epsilon$ and $\|Ta_1 - a_1\| \leq \epsilon$ (3) then $\|Ta_t - a_t\| \leq a(\epsilon)$ and $a(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow +0$.

Proof: Assume that (3) holds with $a_0 \neq a_1$ and $0 < t < 1$. Then let $i = 0, 1$ such that

$$\|a_i - (a_t + Ta_t)/2\| \geq \|a_i - a_t\|$$

If not, would have the contradiction

$$\|a_1 - a_0\| \leq \sum_{i=0}^1 \left\| a_i - \frac{a_t + Ta_t}{2} \right\| < \sum_{i=0}^1 \|a_i - a_t\| = \|a_1 - a_0\|$$

since $a_1 \neq a_0$ we have $r = \|a_t - a_i\| > 0, n = \|a_t - Ta_i\|, m = \|a_i - Ta_t\|$.

Since T is generalized nonexpansive mapping

$$\begin{aligned} \|Ta_t - a_i\| &\leq \|Ta_t - Ta_i\| + \|Ta_i - a_i\| \\ &\leq \delta \|a_t - a_i\| + \mu \left\{ \begin{array}{l} \|a_t - Ta_t\| \\ + \|a_i - Ta_i\| \end{array} \right\} \\ &\quad + \omega \left\{ \begin{array}{l} \|a_t - Ta_i\| \\ + \|a_i - Ta_t\| \end{array} \right\} + \|Ta_i - a_i\| \\ &\leq \delta r + \mu(a(\epsilon) + \epsilon) + \omega(n + m) + \epsilon \end{aligned}$$

let $w = ar + \mu(a(\epsilon) + \epsilon) + \omega(n + m)$. Then $\|Ta_t - a_i\| \leq w + \epsilon$.

Put $a = a_t, b = Ta_t, c = a_i$ and $R = w + \epsilon$.

Let $\eta(\cdot)$ indicate the strictly monotone increasing function to $\zeta(\cdot)$. The diameter of M denotes by $diam(M)$, by theorem (2.10), we have

$$\|Ta_t - a_t\| \leq \sup_{r \in [0, diam(M)]} (w + \epsilon) \eta\left(\frac{\epsilon}{w + \epsilon}\right)$$

the $a(\epsilon)$ defined here has desired properties. First $a(\epsilon) \geq \epsilon \eta(1) = 2\epsilon$ for $w = 0$.

Forming the supremum separately over the two intervals $[0, \sqrt{\epsilon} - \epsilon]$ and monotonicity of $\eta(\cdot)$, that

$$a(\epsilon) \leq \max\{\sqrt{\epsilon} \eta(1), (diam(M) + \epsilon) \eta(\sqrt{\epsilon})\} \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

Since $a(\epsilon) \geq 2\epsilon$, then $\|Ta_t - a_t\| \leq a(\epsilon)$ as $a(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow +0$.

Hence

(3) holds for the remaining cases $a_1 \neq a_0, t = 0, 1$ and $a_0 = a_1$.

Theorem (2.2): Let B be a closed, bounded and convex subset of uniformly convex M , then the operator $I - T$ is demiclosed on B .

Proof: We show that for any sequence (a_n) in M , if (a_n) converges weakly to a and $(I - T)(a_n)$ converges strongly to 0 as $n \rightarrow \infty$, then $a \in M$ and $(I - T)(a) = 0$.

By proposition (1.10), we get $a \in M$.

For $\epsilon_0 \in (0,1)$ choose a sequence (ϵ_n) such that

$$\epsilon_n \leq \epsilon_{n-1} \text{ and } a(\epsilon_n) \leq \epsilon_{n-1}, \forall n \in \mathbb{N}$$

This is possible because $a(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Choosing a subsequence of (a_n) if necessary, we have

$$\|Ta_n - a_n\| \leq \epsilon_n, \forall n \in \mathbb{N}$$

then

$$\|Tb - b\| \leq \epsilon_0, \forall b \in co\{a_n, n \in \mathbb{N}\} \quad (4)$$

Now

i) Let $b_1 \in co\{a_m, a_n\}$ where $1 \leq m < n$, by hypothesis

$$\|Ta_m - a_m\| \leq \epsilon_m \text{ and } \|Ta_n - a_n\| \leq \epsilon_n, \epsilon_n \leq \epsilon_m$$

then

$$\|Tb_1 - b_1\| \leq a(\epsilon_m) \leq \epsilon_{m-1} \leq \epsilon_0$$

ii) Let $b_2 \in co\{a_k, a_m, a_n\}$ where $1 \leq k < m < n$. The key is that $b_2 \in co\{a_k, b_1\}$ since $b_1 \in co\{a_m, a_n\}$ by (i) $\|Tb_1 - b_1\| \leq \epsilon_{m-1}, \epsilon_{m-1} \leq \epsilon_k$, so

$$\|Ta_k - a_k\| \leq \epsilon_k \text{ and } \|Tb_1 - b_1\| \leq \epsilon_k$$

hence

$$\|Tb_2 - b_2\| \leq a(\epsilon_k) \leq \epsilon_{k-1} \leq \epsilon_0$$

if (a_n) converges weakly to a as $n \rightarrow \infty$, then $a \in co\{a_n, n \in \mathbb{N}\}$, by proposition (1.10) and step (4), we obtain

$$\|Ta - a\| \leq \epsilon_0$$

since ϵ_0 can be any arbitrary small, $Ta - a = 0$.

Not only is the map $a \rightarrow Ta$ generalized nonexpansive, but for fixed point b so is $a \rightarrow Ta + b$. This implies that $I - T$ is demiclosed.

Lemma (2.3): Let $T: B \rightarrow B$ be a quasi-nonexpansive map and $S: B \rightarrow B$ be Lipschitzain and generalized nonexpansive maps. Let

i) (a_n) be as in (1) where $(\alpha_n) \in (0,1)$.

ii) (z_n) be as in (2) where (α_n) and $(\beta_n) \in [0,1]$.

If $F(T, S) \neq \emptyset$, then $\lim_{n \rightarrow \infty} \|a_n - a^*\|$ and $\lim_{n \rightarrow \infty} \|z_n - a^*\|$ both exist for all $a^* \in F(T, S)$.

Proof: Let $a^* \in F$.

$$\begin{aligned} \text{i) } \|a_{n+1} - a^*\| &= \|Sb_n - a^*\| \\ &\leq \delta \|b_n - a^*\| + \mu \left\{ \begin{array}{l} \|b_n - Sb_n\| \\ + \|a^* - a^*\| \end{array} \right\} \\ &\quad + \omega \left\{ \begin{array}{l} \|b_n - a^*\| \\ + \|a^* - Sb_n\| \end{array} \right\} \\ &\leq \delta \|b_n - a^*\| + \mu \left\{ \begin{array}{l} \|b_n - a^*\| \\ + \|Sb_n - a^*\| \end{array} \right\} + \\ &\quad \omega \left\{ \begin{array}{l} \|b_n - a^*\| \\ + \|Sb_n - a^*\| \end{array} \right\} \\ &\leq (\delta + \mu + \omega) \|b_n - a^*\| \\ &\leq (\delta + 2\mu + 2\omega) \|b_n - a^*\| \\ &\leq \|b_n - a^*\| \\ &= \|(1 - \alpha_n)a_n + \alpha_n Ta_n - a^*\| \\ &\leq (1 - \alpha_n) \|a_n - a^*\| + \alpha_n \|a_n - a^*\| \\ &= \|a_n - a^*\| \end{aligned}$$

then $\lim_{n \rightarrow \infty} \|a_n - a^*\|$ exists $\forall a^* \in F(T, S)$.

$$\text{ii) } \|u_n - a^*\| \leq (1 - \beta_n) \|Sz_n - a^*\| + \beta_n \|Tz_n - a^*\|$$

$$\leq (1 - \beta_n) \left[\begin{array}{c} \delta \|z_n - a^*\| \\ + \mu \left\{ \begin{array}{c} \|z_n - Sz_n\| \\ + \|a^* - a^*\| \end{array} \right\} + \omega \left\{ \begin{array}{c} \|z_n - a^*\| \\ + \|a^* - Sz_n\| \end{array} \right\} \end{array} \right] + \beta_n \|z_n - a^*\|$$

$$\leq (1 - \beta_n) \left[\begin{array}{c} \delta \|z_n - a^*\| + \mu \left\{ \begin{array}{c} \|z_n - a^*\| \\ + \|Sz_n - a^*\| \end{array} \right\} \\ + \omega \left\{ \begin{array}{c} \|z_n - a^*\| \\ + \|a^* - Sz_n\| \end{array} \right\} \end{array} \right]$$

$$+ \beta_n \|z_n - a^*\| \leq (1 - \beta_n) \left(\begin{array}{c} \delta + \mu + \\ \mu K + \omega + \omega K \\ + \beta_n \|z_n - a^*\| \end{array} \right) \|z_n - a^*\|$$

$$\leq (1 - \beta_n) \left(\begin{array}{c} \delta + 2\mu \\ + 2\omega \end{array} \right) \|z_n - a^*\|$$

$$+ \beta_n \|z_n - a^*\| \leq (1 - \beta_n + \beta_n) \|z_n - a^*\| = \|z_n - a^*\|$$

$$\|z_{n+1} - a^*\| \leq (1 - \alpha_n) \|Sz_n - a^*\| + \alpha_n \|Tu_n - a^*\|$$

$$\leq (1 - \alpha_n) \left[\begin{array}{c} \delta \|z_n - a^*\| + \mu \left\{ \begin{array}{c} \|z_n - Sz_n\| \\ + \|a^* - a^*\| \end{array} \right\} \\ + \omega \left\{ \begin{array}{c} \|z_n - a^*\| \\ + \|a^* - Sz_n\| \end{array} \right\} \end{array} \right]$$

$$+ \alpha_n \|u_n - a^*\| \leq (1 - \alpha_n) \left[\left(\begin{array}{c} \delta + 2\mu \\ + 2\omega \end{array} \right) \|z_n - a^*\| \right]$$

$$+ \alpha_n \|u_n - a^*\| \leq (1 - \alpha_n) \|z_n - a^*\| + \alpha_n \|z_n - a^*\| = \|z_n - a^*\|$$

Then $\lim_{n \rightarrow \infty} \|z_n - a^*\|$ exists $\forall a^* \in F(T, S)$.

Lemma (2.4): Let $T: B \rightarrow B$ be quasi-nonexpansive map and $S: B \rightarrow B$ be Lipschitzain, generalized nonexpansive maps and affine and (a_n) be as in (1). Suppose that the following condition $\|a - Tb\| \leq \|Sa - Tb\|, \forall a, b \in B$ holds. If $F(T, S) \neq \emptyset$, then

$$\lim_{n \rightarrow \infty} \|Ta_n - a_n\| = \lim_{n \rightarrow \infty} \|Sa_n - a_n\| = 0$$

Proof: Let $a^* \in F(T, S)$.

By lemma(2.3.i) $\lim_{n \rightarrow \infty} \|a_n - a^*\|$ exists. Suppose that $\lim_{n \rightarrow \infty} \|a_n - a^*\| = c, \forall c \geq 0$.

If $c = 0$, there is nothing to proof.

Now suppose $c > 0$,

$$\|a_{n+1} - a^*\| = \|Sb_n - a^*\| \leq \|b_n - a^*\|$$

By lemma(2.3.i), we show that $\|b_n - a^*\| \leq \|a_n - a^*\|$

this implies to

$$\limsup_{n \rightarrow \infty} \|b_n - a^*\| \leq c \tag{5}$$

moreover $d = \lim_{n \rightarrow \infty} \|a_{n+1} - a^*\|$

$$\|a_{n+1} - a^*\| \leq \|b_n - a^*\|$$

then

$$c \leq \liminf_{n \rightarrow \infty} \|b_n - a^*\| \tag{6}$$

By (5) and (6), we get

$$\lim_{n \rightarrow \infty} \|b_n - a^*\| = c$$

Next consider

$$c = \|b_n - a^*\| \leq (1 - \alpha_n) \|a_n - a^*\| + \alpha_n \|Ta_n - a^*\|$$

By applying lemma (1.12), we get

$$\lim_{n \rightarrow \infty} \|a_n - Ta_n\| = 0$$

$$c = \lim_{n \rightarrow \infty} \|a_{n+1} - a^*\| = \lim_{n \rightarrow \infty} \|Sb_n - a^*\|$$

$$\|S[(1 - \alpha_n)a_n + \alpha_n Ta_n] - a^*\| \leq (1 - \alpha_n) \|Sa_n - a^*\| + \alpha_n \|STa_n - a^*\|$$

By applying lemma (1.12), we get

$$\lim_{n \rightarrow \infty} \|Sa_n - STa_n\| = 0$$

Now

$$\|Sa_n - a_n\| \leq \|Sa_n - STa_n\| + \|STa_n - a_n\|$$

By using the hypothesis condition, we have

$$\|Sa_n - a_n\| \leq 2\|Sa_n - STa_n\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus

$$\lim_{n \rightarrow \infty} \|Sa_n - a_n\| = 0.$$

Lemma (2.5): Let $T: B \rightarrow B$ be a quasi-nonexpansive map, $S: B \rightarrow B$ be Lipschitzain and generalized nonexpansive maps and (z_n) be as in (2). Suppose that the following condition $\|a - Tb\| \leq \|Sa - Tb\|, \forall a, b \in B$ holds. If $F(T, S) \neq \emptyset$, then

$$\lim_{n \rightarrow \infty} \|Tz_n - z_n\| = \lim_{n \rightarrow \infty} \|Sz_n - z_n\| = 0.$$

Proof: Let $a^* \in F(T, S)$.

By lemma (2.3.ii) $\lim_{n \rightarrow \infty} \|z_n - a^*\|$ exists. Suppose

$$\text{that } \lim_{n \rightarrow \infty} \|z_n - a^*\| = c, \forall c \geq 0.$$

If $c = 0$, there is nothing to proof.

Now suppose $c > 0$

$$\lim_{n \rightarrow \infty} \|z_{n+1} - a^*\| = c$$

$$c = \|z_{n+1} - a^*\| \leq (1 - \alpha_n) \|Sz_n - a^*\| + \alpha_n \|Tu_n - a^*\|$$

By applying lemma (1.12), we get

$$\lim_{n \rightarrow \infty} \|Sz_n - Tu_n\| = 0$$

$$\|a_{n+1} - a^*\| = \|(1 - \alpha_n)Sz_n + \alpha_n Tu_n - a^*\| \leq \|Sz_n - a^*\| + \alpha_n \|Sz_n - Tu_n\|$$

this implies to

$$c \leq \liminf_{n \rightarrow \infty} \|Sz_n - a^*\| \tag{7}$$

$$\text{and } \|Sz_n - a^*\| \leq \|z_n - a^*\|$$

therefore

$$\limsup_{n \rightarrow \infty} \|Sz_n - a^*\| \leq c \tag{8}$$

By (7) and (8), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|S z_n - a^*\| &= c \\ \|S z_n - a^*\| &\leq \|S z_n - T u_n\| + \|T u_n - a^*\| \\ \text{that yields to} \\ c &\leq \liminf_{n \rightarrow \infty} \|u_n - a^*\| \end{aligned} \tag{9}$$

and

$$\begin{aligned} \|u_n - a^*\| &\leq (1 - \beta_n) \|S z_n - a^*\| + \beta_n \|T z_n - a^*\| \\ &= \|z_n - a^*\| \\ \text{Now} \\ \limsup_{n \rightarrow \infty} \|u_n - a^*\| &\leq c \end{aligned} \tag{10}$$

By (9) and (10), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|u_n - a^*\| &= c \\ c &= \|u_n - a^*\| \\ &\leq (1 - \beta_n) \|S z_n - a^*\| + \beta_n \|T z_n - a^*\| \end{aligned}$$

By applying lemma (1.12), we obtain

$$\lim_{n \rightarrow \infty} \|S z_n - T z_n\| = 0$$

Now

$$\|S z_n - z_n\| \leq \|S z_n - T z_n\| + \|T z_n - z_n\|$$

By using the hypothesis condition, we get

$$\|S z_n - z_n\| \leq 2 \|S z_n - T z_n\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

and

$$\begin{aligned} \|T z_n - z_n\| &\leq \|T z_n - S z_n\| + \|S z_n - z_n\| \\ &\leq 2 \|T z_n - S z_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence

$$\lim_{n \rightarrow \infty} \|T z_n - z_n\| = \lim_{n \rightarrow \infty} \|S z_n - z_n\| = 0.$$

Lemma (2.6): Let $T: B \rightarrow B$ be Lipschitzain and quasi-nonexpansive maps and $S: B \rightarrow B$ be lipschitzain and generalized nonexpansive maps. Then for $a_1^*, a_2^* \in F(T, S)$, (a_n) be as in (1) and (z_n) be as in (2) such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|t a_n + (1 - t) a_1^* - a_2^*\| &\text{ and} \\ \lim_{n \rightarrow \infty} \|t z_n + (1 - t) a_1^* - a_2^*\| &= 0, \forall t \in [0, 1]. \end{aligned}$$

Proof: Now to prove $\lim_{n \rightarrow \infty} \|t a_n + (1 - t) a_1^* - a_2^*\|$ exists and equal to zero, by lemma (2.3.i) $\lim_{n \rightarrow \infty} \|a_n - a^*\|$ exists, $\forall a^* \in F(T, S)$ and (a_n) is bounded.

Then there is a real number $L > 0$ such that $(a_n) \subseteq D = \overline{B_r(0)} \cap B$, so that $D \neq \emptyset$ is a closed convex bounded subest of B.

Put $\gamma_n(t) = \|t a_n + (1 - t) a_1^* - a_2^*\|$. Notice that $\gamma_n(0) = \|a_1^* - a_2^*\|$ and $\gamma_n(1) = \|a_n - a_2^*\|$ esists by lemma(2.3.i).

$$\begin{aligned} \text{Defin } R_n: D &\rightarrow D, \forall n \in N, R_n a = \\ &S[(1 - \alpha_n) a_n + \alpha_n T a_n] \forall a \in D. \\ \|R_n a - R_n b\| &= \left\| S[(1 - \alpha_n) a_n + \alpha_n T a_n] \right\| \\ &= \left\| -S[(1 - \alpha_n) b_n + \alpha_n T b_n] \right\| \\ &\leq (1 - \alpha_n) \|a_n - b_n\| + \alpha_n \|T a_n - T b_n\| \end{aligned}$$

$$\begin{aligned} &\leq (1 - \alpha_n) \|a_n - b_n\| + \alpha_n \|T a_n - a^*\| \\ &\quad + \alpha_n \|T b_n - a^*\| \\ &\leq (1 - \alpha_n) \|a_n - a^*\| + (1 - \alpha_n) \|b_n - a^*\| \\ &\quad + \alpha_n \|a_n - a^*\| + \alpha_n \|b_n - a^*\| \\ &= \|a_n - a^*\| + \|b_n - a^*\| \end{aligned}$$

Set $W_{n,m} = R_{n+m} R_{n+m-1} \dots R_n$ and $b_{n,m} = \|W_{n,m}(t a_n + (1 - t) a_1^*) - (t W_{n,m} a_n + (1 - t) a_1^*)\|, \forall n, m \in N$.

Then

$$\begin{aligned} \|W_{n,m} a - W_{n,m} b\| &\leq \|W_{n,m} a - a^*\| + \|W_{n,m} b - a^*\| \\ &\leq \|a - a^*\| + \|b - a^*\| \\ \text{and } \|W_{n,m} a - a^*\| &\leq \|a - a^*\|, W_{n,m} a_n = \\ &a_{n+m} \text{ and } W_{n,m} a^* = a^*, \forall a^* \in F. \end{aligned}$$

By lemma(1.13) there is a strictly increasing function continuous function $f: [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ such that

$$\begin{aligned} b_{n,m} &\leq K f^{-1} \left(\|a_n - a_1^*\| - \frac{1}{K} \left\| \begin{matrix} W_{n,m} a_n \\ -W_{n,m} a_1^* \end{matrix} \right\| \right) \\ &\leq K f^{-1} \left(\|a_n - a_1^*\| - \frac{1}{K} \|a_{n+m} - a_1^*\| \right) \end{aligned}$$

since $\lim_{n \rightarrow \infty} \|a_n - a^*\|$ exists $\forall a^* \in F$.

$$\lim_{n \rightarrow \infty} \sup f(b_{n,m}) = 0 \xrightarrow{\text{yields}} \lim_{n \rightarrow \infty} \limsup_{m \rightarrow \infty} b_{n,m} = 0$$

Now,

$$\begin{aligned} \gamma_{n+m}(t) &= \|t a_{n+m} + (1 - t) a_1^* - a_2^*\| \\ &= \|t W_{n,m} a_n + (1 - t) a_1^* - a_2^*\| \end{aligned}$$

$$= \left\| \begin{matrix} t W_{n,m} a_n + (1 - t) a_1^* - a_2^* + \\ W_{n,m}(t a_n + (1 - t) a_1^*) \\ - a_2^* + a^* - a^* \\ -W_{n,m}(t a_n + (1 - t) a_1^*) + a_2^* \end{matrix} \right\|$$

$$\begin{aligned} &\leq b_{n,m} + \|W_{n,m}(t a_n + (1 - t) a_1^*) - a_2^*\| \\ &\leq b_{n,m} + \|W_{n,m}(t a_n + (1 - t) a_1^*) - W_{n,m} a_2^*\| \\ &\leq b_{n,m} + \gamma_n(t) \end{aligned}$$

Now

$$\lim_{n \rightarrow \infty} \sup \gamma_{n+m}(t) \leq \lim_{n \rightarrow \infty} \sup b_{n,m} + \gamma_n(t)$$

then

$$\lim_{n \rightarrow \infty} \sup \gamma_{n+m}(t) \leq \lim_{n \rightarrow \infty} \inf \gamma_n(t)$$

which implies that $\lim_{n \rightarrow \infty} \|t a_n + (1 - t) a_1^* - a_2^*\|$ exists $\forall t \in [0, 1]$.

Now to prove $\lim_{n \rightarrow \infty} \|t z_n + (1 - t) a_1^* - a_2^*\|$ exists.

By lemma (2.3.ii) $\lim_{n \rightarrow \infty} \|z_n - a^*\|$ exists, $\forall a^* \in F(T, S)$ and (z_n) is bounded.

Then there is a real number $L > 0$ such that $(z_n) \subseteq D = \overline{B_r(0)} \cap B$ so that $D \neq \emptyset$ is a closed convex bounded subest of B.

Put $\gamma_n(t) = \|tz_n + (1-t)a_1^* - a_2^*\|$
 notice that $\gamma_n(0) = \|a_1^* - a_2^*\|$ and $\gamma_n(1) = \|a_n - a_2^*\|$ exists by lemma (2.3.ii).

Define $R_n: D \rightarrow D, \forall n \in N, R_n z = (1 - \alpha_n)Sz_n + \alpha_n T((1 - \beta_n)Sz_n + \beta_n Tz_n)$

$$\begin{aligned} & \|R_n z - R_n w\| \\ &= \left\| \begin{matrix} (1 - \alpha_n)Sz_n + \alpha_n Tu_n \\ -(1 - \alpha_n)Sw_n - \alpha_n Tv_n \end{matrix} \right\| \\ &\leq (1 - \alpha_n)\|Sz_n - Sw_n\| + \alpha_n\|Tu_n - Tv_n\| \\ &\leq (1 - \alpha_n)\left(\begin{matrix} \delta + 2\mu \\ +2\omega \end{matrix}\right)\|z_n - w_n\| \\ &\quad + \alpha_n\|Tu_n - a^*\| + \alpha_n\|Tv_n - a^*\| \\ &\leq (1 - \alpha_n)\|z_n - w_n\| + \alpha_n\|u_n - a^*\| \\ &\quad + \alpha_n\|v_n - a^*\| \\ &\leq (1 - \alpha_n)\|z_n - w_n\| \\ &+ \alpha_n \left\{ \begin{matrix} (1 - \beta_n)\|z_n - a^*\| \\ + \beta_n\|z_n - a^*\| \end{matrix} \right\} \\ &+ \alpha_n \left\{ \begin{matrix} (1 - \beta_n)\|w_n - a^*\| \\ + \beta_n\|w_n - a^*\| \end{matrix} \right\} \\ &= \|z_n - a^*\| + \|w_n - a^*\| \end{aligned}$$

The rest of the proof follows the pattern of the above argument.

Theorem (2.7): Let M be a uniformly convex Banach space satisfying Opial's condition and $T: B \rightarrow B$ be quasi-nonexpansive map with $(I - T)$ demiclosed at zero, $S: B \rightarrow B$ be Lipschitzain and generalized nonexpansive maps and $(a_n), (z_n)$ as in lemma (2.4) and lemma (2.5), respectively. If $F(T, S) \neq \emptyset$, then (a_n) and (z_n) both converge weakly to a common fixed point of S and T .

Proof: Let $a^* \in F(T, S)$. As proved in lemma (2.3) $\lim_{n \rightarrow \infty} \|a_n - a^*\|$ and $\lim_{n \rightarrow \infty} \|z_n - a^*\|$ exist.

Now, must prove that (a_n) converges weakly to a unique weak subsequential limit in F .

Since (a_n) is bounded sequence in M , there exist two convergent subsequences (a_{ni}) and (a_{nj}) of (a_n) .

Let $x_1, x_2 \in B$ be weak limit of (a_{ni}) and (a_{nj}) respectively. By lemma (2.4) $\lim_{n \rightarrow \infty} \|Sa_n - a_n\| = 0$.

By proposition (2.1) and theorem (2.2), we get $I - S$ is demiclosed to zero.

Then $Se_1 = e_1$ and by hypothesis $I - T$ is demiclosed so, $Te_1 = e_1$. In the same way, can prove that $e_2 \in F(T, S)$.

To prove the uniqueness, assume $e_1 \neq e_2$. Then by Opial's condition:

$$\begin{aligned} \lim_{n \rightarrow \infty} \|a_n - e_1\| &= \lim_{n \rightarrow \infty} \|a_{ni} - e_1\| \\ &< \lim_{n \rightarrow \infty} \|a_n - e_2\| \\ &= \lim_{n \rightarrow \infty} \|a_{nj} - e_2\| \\ &< \lim_{n \rightarrow \infty} \|a_{nj} - e_1\| \end{aligned}$$

$$= \lim_{n \rightarrow \infty} \|a_n - e_1\|$$

this is contradiction. Thus (a_n) converges weakly to a point in $F(T, S)$.

By utilizing the same above argument, we can prove that (z_n) converges weakly to a point in $F(T, S)$.

Theorem (2.8): Let M be a uniformly convex Banach space such that its dual M^* satisfies the Kadec-Klee property. Let $T, S, B, (a_n)$ and (z_n) be as in lemma (2.4) and lemma (2.5), respectively. If $F(T, S) \neq \emptyset$, then (a_n) and (z_n) converge weakly to a common fixed point of S and T .

Proof: Since (a_n) and (z_n) are bounded and M is reflexive. Then, there is a subsequence (a_{ni}) of (a_n) that converges weakly to a point $a^* \in B$. By lemma (2.4)

$$\lim_{n \rightarrow \infty} \|Sa_{ni} - a_{ni}\| = 0 = \lim_{n \rightarrow \infty} \|Ta_{ni} - a_{ni}\|$$

thus $a^* \in F(T, S)$.

To prove (a_n) converges weakly to a point a^* .

Assume that (a_{nk}) is another subsequence of (a_n) that converges weakly to a point $b^* \in B$.

Then by lemma (2.6) $\lim_{n \rightarrow \infty} \|ta_n + (1-t)a^* - b^*\|$ exists $\forall t \in [0, 1]$.

By lemma (1.15) $a^* = b^*$. Then (a_n) converges weakly to the point $a^* \in F(T, S)$.

Utilizing the same above argument to prove that (z_n) converges weakly to the point $a^* \in F(T, S)$.

The following corollary as a special case of quasi-nonexpansive mapping is now obvious.

Corollary (2.8): Let M be a uniformly convex Banach space satisfying Opial's condition and $T: B \rightarrow B$ be satisfying condition (C_λ) , $S: B \rightarrow B$ be generalized nonexpansive map and $(a_n), (z_n)$ be as in lemma (2.4) and lemma (2.5), respectively. If $F(T, S) \neq \emptyset$, then (a_n) and (z_n) converges weakly to a common fixed point of S and T .

Corollary (2.9): Let M be a uniformly convex Banach space and its dual M^* satisfies the Kadec-Klee property and $T: B \rightarrow B$ be Lipschitzain map and satisfying condition (C_λ) and $S: B \rightarrow B$ be Lipschitzain and generalized nonexpansive maps and $(a_n), (z_n)$ be as in lemma (2.6). If $F(T, S) \neq \emptyset$, then (a_n) and (z_n) converges weakly to a common fixed point of S and T .

Corollary (2.10): Let M be a uniformly convex Banach space satisfying Opial's condition and $T: B \rightarrow B$ be satisfying condition (E_λ) , $S: B \rightarrow B$ be generalized nonexpansive map and $(a_n), (z_n)$ be as in lemma (2.4) and lemma (2.5), respectively. If $F(T, S) \neq \emptyset$, then (a_n) and (z_n)

converges weakly to a common fixed point of S and T.

Corollary (2.11): Let M be a uniformly convex Banach space and its dual M^* satisfies the Kadec-Klee property and $T: B \rightarrow B$ be Lipschitzian map and satisfying condition (E_λ) and $S: B \rightarrow B$ be Lipschitzian and generalized nonexpansive maps and $(a_n), (z_n)$ be as in lemma (2.6). If $F(T, S) \neq \emptyset$, then (a_n) and (z_n) converges weakly to a common fixed point of S and T.

3. Equivalence of Iterations

Theorem (3.1): Let B be a nonempty closed convex subset of a Banach space M . Let $T: B \rightarrow B$ be a quasi-nonexpansive map, $S: B \rightarrow B$ be Lipschitzian and generalized nonexpansive maps and $a^* \in B$ be a common fixed point of S and T. Let (a_n) and (z_n) be the Picard-Mann and Liu et al iteration schemes defined in (1) and (2), respectively. Suppose (α_n) and (β_n) satisfied the following conditions:

- 1- (α_n) and $(\beta_n) \in (0, 1), \forall n \geq 0$.
- 2- $\sum \alpha_n = \infty$.
- 3- $\sum \alpha_n \beta_n < \infty$.

If $z_0 = a_0$ and $R(T), R(S)$ are bounded, then the Picard-Mann iterative sequence (a_n) converges strongly to a^* ($a_n \rightarrow a^*$) and the Liu et al iterative sequence (z_n) converges strongly to a^* ($z_n \rightarrow a^*$).

Proof: Since the range of T and S are bounded, let $M = \sup_{a \in B} \{ \|Ta\| \} + \|a_0\| < \infty$

then

$$\|a_n\| \leq M, \|b_n\| \leq M, \|z_n\| \leq M, \|u_n\| \leq M$$

therefore

$$\begin{aligned} & \|Ta_n\| \leq M, \|Tz_n\| \leq M \\ & \|a_{n+1} - z_{n+1}\| \\ &= \|Sb_n - (1 - \alpha_n)Sz_n - \alpha_n Tu_n\| \\ &\leq \|Sb_n - Sz_n\| + \alpha_n \|Sz_n - Tu_n\| \\ &\leq \|Sb_n - a^*\| + \|Sz_n - a^*\| + \alpha_n \|Sz_n - a^*\| \\ &\quad + \alpha_n \|Tu_n - a^*\| \\ &\leq \delta \|b_n - a^*\| + \mu \left\{ \frac{\|b_n - Sb_n\|}{+\|a^* - a^*\|} \right\} \\ &+ \omega \left\{ \frac{\|b_n - a^*\|}{+\|a^* - Sb_n\|} \right\} + \delta \|z_n - a^*\| \\ &+ \mu \left\{ \frac{\|b_n - Sb_n\|}{+\|a^* - a^*\|} \right\} + \omega \left\{ \frac{\|b_n - a^*\|}{+\|a^* - Sb_n\|} \right\} \\ &+ \alpha_n \|z_n - a^*\| + \alpha_n \|u_n - a^*\| \\ &\leq \left(\frac{\delta + 2\mu}{+2\omega} \right) \|b_n - a^*\| + \left(\frac{\delta + 2\mu}{+2\omega} \right) \|z_n - a^*\| \\ &\quad + \alpha_n \|z_n - a^*\| + \alpha_n \|u_n - a^*\| \\ &\leq \|b_n - a^*\| + (1 + \alpha_n) \|z_n - a^*\| \\ &\quad + \alpha_n \|u_n - a^*\| \end{aligned}$$

$$\begin{aligned} & \|b_n - a^*\| \\ &\leq (1 - \alpha_n) \|a_n - a^*\| + \alpha_n \|Ta_n - a^*\| \\ &\leq (1 - \alpha_n) \|a_n - a^*\| + \alpha_n \{ \|Ta_n\| + \|a^*\| \} \end{aligned}$$

$$\begin{aligned} & \leq (1 - \alpha_n) \|a_n - a^*\| + \alpha_n \|M + \|a^*\|\| \\ & \|u_n - a^*\| \\ & \leq (1 - \beta_n) \|Sz_n - a^*\| + \beta_n \|Tz_n - a^*\| \\ & \leq (1 - \beta_n) \left(\frac{\delta + 2\mu}{+2\omega} \right) \|z_n - a^*\| + \beta_n \left\{ \frac{\|Tz_n\|}{+\|a^*\|} \right\} \\ & \leq (1 - \beta_n) \{M + \|a^*\|\} + \beta_n \{M + \|a^*\|\} \\ & = M + \|a^*\| \end{aligned}$$

Thus

$$\begin{aligned} & \|a_{n+1} - z_{n+1}\| \\ & \leq \|b_n - a^*\| + (1 + \alpha_n) \|z_n - a^*\| \\ & \quad + \alpha_n \|u_n - a^*\| \\ & \leq (1 - \alpha_n) \|a_n - a^*\| + \alpha_n \{M + \|a^*\|\} \\ & \quad + (1 + \alpha_n) \|z_n - a^*\| + \alpha_n \{M + \|a^*\|\} \\ & \leq (1 - \alpha_n) \|a_n - z_n\| + (1 - \alpha_n) \|z_n - a^*\| \\ & \quad + 2\alpha_n \{M + \|a^*\|\} + (1 + \alpha_n) \|z_n - a^*\| \\ & \leq (1 - \alpha_n) \|a_n - z_n\| + \|z_n - a^*\| + 2\alpha_n \\ & \quad \{M + \|a^*\|\} \\ & \leq (1 - \alpha_n) \|a_n - z_n\| + (1 + 2\alpha_n) \{M + \|a^*\|\} \end{aligned}$$

let $\mu_n = \|a_n - z_n\|, \rho_n = (1 + 2\alpha_n) \{M + \|a^*\|\}, \sigma_n = \alpha_n$ and $\frac{\rho_n}{\sigma_n} \rightarrow 0$ as $n \rightarrow \infty$.

By applying lemma(1.11), we get

$$\lim_{n \rightarrow \infty} \|a_n - z_n\| = 0.$$

If $a_n \rightarrow a^* \in F(T, S)$, then

$$\|z_n - a^*\| \leq \|z_n - a_n\| + \|a_n - a^*\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

And if $z_n \rightarrow a^* \in F(T, S)$, then

$$\|a_n - a^*\| \leq \|a_n - z_n\| + \|z_n - a^*\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

4. Numerical examples

In this section, we consider two examples to show that the Picard-Mann iteration scheme converges faster than Liu et al iteration scheme.

Example (4.1): Let $T, S: R \rightarrow R$ be a map defined by $Ta = \frac{2a}{3}$ and $Sa = \frac{a}{2}, \forall a \in R$. Choose $\alpha_n = \beta_n = \frac{3}{4}, \forall n$ with initial value $a_1 = 30$. The two iteration scheme converge to the same fixed point $a^* = 0$. It's clear from table 1, that Picard-Mann converges faster than Liu et al.

Table 1: Numerical results corresponding to $a_1 = 30$ for 20 steps

n	Iterati on (1)	Iterati on (2)	n	Iterati on (1)	Iterati on (2)
0	30	20	1	0.000	0.003
			1	6	4
1	11.2500	13.1250	1	0.000	0.001
			2	2	5

2	4.218	5.742	1	0.000	0.000
	8	2	3	1	6
3	1.582	2.512	1	0.000	0.000
	0	2	4	0	3
4	0.593	1.099	1	0.000	0.000
	3	1	5	0	1
5	0.222	0.480	1	-	0.000
	5	9	6	-	1
6	0.083	0.210	1	-	0.000
	4	4	7	-	0
7	0.031	0.092	1	-	0.000
	3	0	8	-	0
8	0.011	0.040	1	-	0.000
	7	3	9	-	0
9	0.004	0.017	2	-	0.000
	4	6	0	-	0
1	0.001	0.007	-	-	0.000
0	6	7	-	-	0

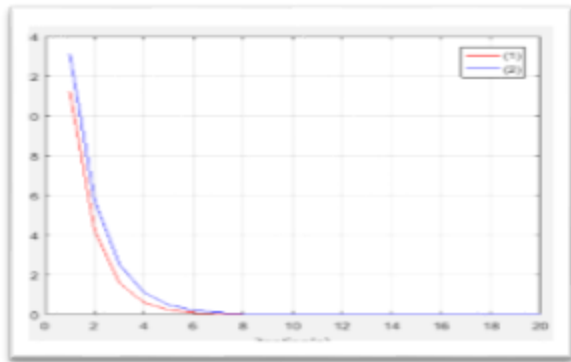


Figure 1: Convergence behavior corresponding to $a_1 = 30$ for 20 steps.

Example (4.2): Let $B = [-180,180]$, $T, S: B \rightarrow B$ be a map defined by $Ta = a \cos a$ and $Sa = \frac{a}{2} \forall a \in B$. Choose $\alpha_n = \frac{1}{3}, \beta_n = \frac{1}{5} \forall n$ with initial value $a_1 = 30$. The two iteration scheme converge to the same fixed point $a^* = 0$. It's clear from table 2, that Picard-Mann converges faster than Liu et al.

Table 2: Numerical results corresponding to $a_1 = 30$ for 30 steps

n	Iterati on (1)	Iterati on (2)	n	Iterati on (1)	Iterati on (2)
0	30	30	1	0.0001	0.0014
			6		
1	10.7713	12.9255	1	0.0000	0.0008
			7		
2	3.1911	7.5904	1	0.0000	0.0005
			8		
3	0.5325	3.4317	1	0.0000	0.0003
			9		
4	0.2540	0.7150	2	0.0000	0.0002
			0		
5	0.1256	0.3940	2	-	0.0001
			1		
6	0.0626	0.2304	2	-	0.0001
			2		
7	0.0313	0.1370	2	-	0.0000
			3		
8	0.0156	0.0819	2	-	0.0000
			4		
1	0.0078	0.0491	2	-	-
			5		
1	0.0039	0.0295	2	-	-
			6		
1	0.0020	0.0177	2	-	-
			7		
1	0.0010	0.0106	2	-	-
			8		
1	0.0005	0.0064	2	-	-
			9		
1	0.0002	0.0038	3	-	-
			0		
1	0.0001	0.0023			
			5		

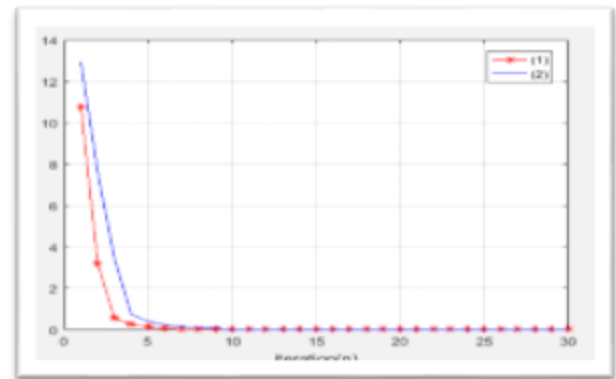


Figure 2: Convergence behavior corresponding to $a_1 = 30$ for 30 steps.

Finally, it is appropriate to ask a question about the possibility of employing the above results in finding solutions to problems such in [19] and [20]

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